HEIGHT OF EXCEPTIONAL COLLECTIONS AND HOCHSCHILD COHOMOLOGY OF QUASIPHANTOM CATEGORIES

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Abstract. We define the normal Hochschild cohomology of an admissible subcategory of the derived category of coherent sheaves on a smooth projective variety X — a graded vector space which controls the restriction morphism from the Hochschild cohomology of X to the Hochschild cohomology of the orthogonal complement of this admissible subcategory. When the subcategory is generated by an exceptional collection, we define its new invariant (the height) and show that the orthogonal to an exceptional collection of height h in the derived category of a smooth projective variety X has the same Hochschild cohomology as X in degrees up to $h-2$. We use this to describe the second Hochschild cohomology of quasiphantom categories in the derived categories of some surfaces of general type. We also give necessary and sufficient conditions of fullness of an exceptional collection in terms of its height and of its normal Hochschild cohomology.

1. Introduction

Assume X is a smooth projective variety and $\mathbf{D}^b(\mathsf{coh}(X)) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition. The goal of the present paper is to describe the Hochschild cohomology of $\mathcal{A}$ and the restriction morphism $\mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A})$ in terms of the category $\mathcal{B}$, especially in case when $\mathcal{B}$ is generated by an exceptional collection.

This question is motivated by the investigation of so-called quasiphantom categories. A quasiphantom category $\mathcal{A}$ is a semiorthogonal component of the bounded derived category of coherent sheaves on a smooth projective variety which has trivial Hochschild homology. Recently, several examples of such categories with X being a surface of general type have been constructed: when X is the classical Godeaux surface a quasiphantom was constructed by Böhning–Graf von Bothmer–Sosna [BBS], when X is a Burniat surface — by Alexeev–Orlov [AO], when X is the Beauville surface — by Galkin–Shinder [GS], and when X is a determinantal Barlow surface — by Böhning–Graf von Bothmer–Katzarkov–Sosna [BBKS].

In all these examples the quasiphantom is the orthogonal complement of an exceptional collection.

The structure of quasiphantom categories is very interesting but little understood as yet. In particular, no direct way to compute their invariants such as Hochschild cohomology is known. So, it is natural to do this using the information from the given semiorthogonal decomposition. This leads to the question formulated in the first paragraph.

The first result of the paper answers this question in the most general form. Let $\mathscr{D}$ be a smooth and proper DG-category and $\mathscr{B} \subset \mathscr{D}$ its DG-subcategory. We define the normal Hochschild cohomology of $\mathscr{B}$ in $\mathscr{D}$ as the derived tensor product of DG-bimodules

$$\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D}) = \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}^{\mathsf{opp}} \otimes \mathscr{B}} \mathscr{D}^\vee_{\mathscr{B}},$$

where the first factor of the above tensor product is the diagonal $\mathscr{B}$-bimodule while in the second factor $\mathscr{D}^\vee = \mathsf{RHom}_{\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})}(\mathscr{D}, \mathscr{D} \otimes_{\mathsf{k}} \mathscr{D})$ is the dual of the diagonal $\mathscr{D}$-bimodule and $\mathscr{D}^\vee_{\mathscr{B}}$ is its restriction to $\mathscr{B}$.

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We show that if X is a smooth projective variety, $\mathscr{D}$ is a pretriangulated enhancement of $\mathbf{D}(X)$, the unbounded derived category of quasicoherent sheaves on X, and $\mathscr{B} \subset \mathscr{D}$ the induced enhancement of the semiorthogonal component $\mathcal{B} \subset \mathbf{D}^b(\mathsf{coh}(X))$ then there is a distinguished triangle

$$\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D}) \to \mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A}).$$

We define the *height* of the subcategory $\mathcal{B}$ as the minimal integer h such that $\mathsf{NHH}^h(\mathscr{B}, \mathscr{D}) \neq 0$. It follows immediately that for $t \le h-2$ the restriction morphism $\mathsf{HH}^t(X) \to \mathsf{HH}^t(\mathcal{A})$ is an isomorphism and for $t = h-1$ it is a monomorphism.

Of course, a reasonable way of computing the height (and the normal Hochschild cohomology) is required. In case when the subcategory $\mathcal{B}$ is generated by an exceptional collection $E_1, \dots, E_n$, we construct a spectral sequence which computes $\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D})$ in terms of Ext-groups $\mathsf{Ext}^\bullet(E_i, E_j)$ and $\mathsf{Ext}^\bullet(E_i, S^{-1}(E_j))$, where $S^{-1}(F) = F \otimes \omega_X^{-1}[-\dim X]$ is the inverse Serre functor. The differentials in the spectral sequence are expressed in terms of the Yoneda product and higher multiplications in the natural A_∞ structure.

Of course, usually it is not easy to control higher multiplications, so explicit computation of the height may be difficult. So, we define the *pseudoheight* of an exceptional collection $E_1, \dots, E_n$ as the minimal integer h such that the first page of the above spectral sequence has a nontrivial term in degree h. By definition, the pseudoheight bounds the height from below, and thus controls the restriction morphism of Hochschild cohomology as well.

We illustrate the computation of the height and of the pseudoheight by considering the quasiphantoms in the classical Godeaux, Burniat and the Beauville surfaces. We show that in all cases the height of the collections is 4, while the pseudoheight varies from 4 to 3 depending on the particular case. It follows that the restriction morphism $\mathsf{HH}^t(X) \to \mathsf{HH}^t(\mathcal{A})$ is an isomorphism for $t \le 2$ and a monomorphism for $t = 3$ in all these cases. We also deduce from this the fact that the formal deformation spaces of all considered surfaces X are isomorphic to the formal deformation spaces of the quasiphantom subcategories.

Finally, we show that the height (and the pseudoheight) can be used to verify whether a given exceptional collection is full. On one hand, if the height is strictly positive, one can easily deduce that the collection is not full. On the other hand, we give a sufficient condition of fullness of an exceptional collection which uses the explicit complex computing the height of an exceptional collection and seems to be related to quantum determinants considered by Bondal and Polishchuk.

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2. Preliminaries

2.1. **DG-categories.** For a detailed survey of DG-categories we refer to [Ke] and references therein. Here we only sketch some basic definitions.

A DG-category over a field k is a category $\mathscr{D}$ such that for any pair of objects $\mathbf{x}, \mathbf{y} \in \mathscr{D}$ the set of morphisms $\mathsf{Hom}_{\mathscr{D}}(\mathbf{x}, \mathbf{y})$ comes with a structure of a chain complex of k-vector spaces such that for each object $\mathbf{x}$ the identity morphism $\mathsf{id}_{\mathbf{x}} \in \mathsf{Hom}_{\mathscr{D}}(\mathbf{x}, \mathbf{x})$ is closed of degree zero and the composition map $\mathsf{Hom}_{\mathscr{D}}(\mathbf{x}, \mathbf{y}) \otimes \mathsf{Hom}_{\mathscr{D}}(\mathbf{y}, \mathbf{z}) \to \mathsf{Hom}_{\mathscr{D}}(\mathbf{x}, \mathbf{z})$ is a morphism of complexes (the Leibniz rule). The simplest example of a DG-category is the category $\mathsf{dgm}\text{-}\mathsf{k}$ of complexes of k-vector spaces with

$$\mathsf{Hom}_{\mathsf{dgm}\text{-}\mathsf{k}}(M^\bullet, N^\bullet)^t = \prod_{i \in \mathbb{Z}} \mathsf{Hom}(M^i, N^{i+t}), \qquad d^t(f^i) = (d_N^{i+t} \circ f^i - (-1)^t f^{i+1} \circ d_M^i).$$

The $\mathsf{homotopy\ category}$ $[\mathscr{D}]$ of a DG-category $\mathscr{D}$ is defined as the category which has the same objects as $\mathscr{D}$ and with

$$\mathsf{Hom}_{[\mathscr{D}]}(\mathbf{x},\mathbf{y}) = H^0(\mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{y})).$$

For example, the homotopy category $[\mathsf{dgm}\text{-}\mathsf{k}]$ is the category of complexes with morphisms being chain morphisms of complexes up to a homotopy.

A $\mathsf{DG\text{-}functor}$ $F : \mathscr{D} \to \mathscr{D}'$ is a k-linear functor such that for any pair of objects $\mathbf{x}, \mathbf{y} \in \mathscr{D}$ the morphism $F : \mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{y}) \to \mathsf{Hom}_{\mathscr{D}'}(F(\mathbf{x}), F(\mathbf{y}))$ is a morphism of complexes. If $\mathscr{D}$ is a small DG-category (objects form a set) then DG-functors from $\mathscr{D}$ to $\mathscr{D}'$ also form a DG-category with

$$\mathsf{Hom}(F,G) = \mathsf{Ker}\left(\prod_{\mathbf{x}\in\mathscr{D}} \mathsf{Hom}_{\mathscr{D}'}(F(\mathbf{x}), G(\mathbf{x})) \to \prod_{\mathbf{x},\mathbf{y}\in\mathscr{D}} \mathsf{Hom}(\mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{y}), \mathsf{Hom}_{\mathscr{D}'}(F(\mathbf{y}), G(\mathbf{x})))\right)$$

A $\mathsf{right\ DG\text{-}module}$ over $\mathscr{D}$ is a DG-functor from $\mathscr{D}^{\mathsf{opp}}$, the opposite DG-category, to $\mathsf{dgm}\text{-}\mathsf{k}$. A $\mathsf{left\ DG\text{-}module}$ over $\mathscr{D}$ is a DG-functor from $\mathscr{D}$ to $\mathsf{dgm}\text{-}\mathsf{k}$. The DG-category of right (resp. left) DG-modules over $\mathscr{D}$ is denoted by $\mathsf{dgm}\text{-}\mathscr{D}$ (resp. $\mathsf{dgm}\text{-}\mathscr{D}^{\mathsf{opp}}$). The homotopy category $[\mathsf{dgm}\text{-}\mathscr{D}]$ of DG-modules has a natural triangulated structure. Moreover, it has arbitrary direct sums.

The $\mathsf{Yoneda\ DG\text{-}functor}$ $h : \mathscr{D} \to \mathsf{dgm}\text{-}\mathscr{D}$ is defined by

$$\mathbf{x} \mapsto h^{\mathbf{x}}(\mathbf{y}) = \mathsf{Hom}_{\mathscr{D}}(\mathbf{y},\mathbf{x}).$$

The Yoneda DG-functor for the opposite DG-category can be written as $\mathbf{x} \mapsto h_{\mathbf{x}}(\mathbf{y}) = \mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{y})$. The Yoneda DG-functors are full and faithful. Moreover, one has

$$\mathsf{Hom}_{\mathsf{dgm}\text{-}\mathscr{D}}(h^{\mathbf{x}}, M) = M(\mathbf{x}), \qquad \mathsf{Hom}_{\mathsf{dgm}\text{-}\mathscr{D}^{\mathsf{opp}}}(h_{\mathbf{x}}, N) = N(\mathbf{x})$$

for any right $\mathscr{D}$-module M and any left $\mathscr{D}$-module N. The DG-modules in the images of the Yoneda functors are called $\mathsf{representable}$. The minimal triangulated subcategory of $[\mathsf{dgm}\text{-}\mathscr{D}]$ containing all representable DG-modules and closed under direct summands is called $\mathsf{the\ category\ of\ perfect\ DG\text{-}modules\ over}$ $\mathscr{D}$ and is denoted by $\mathsf{Perf}(\mathscr{D})$. Its objects are called $\mathsf{perfect}$ DG-modules.

A DG-module M is $\mathsf{acyclic}$ if for each object $\mathbf{x} \in \mathscr{D}$ the complex $M(\mathbf{x}) \in \mathsf{dgm}\text{-}\mathsf{k}$ is acyclic. The DG-category of acyclic DG-modules is denoted by $\mathsf{acycl}\text{-}\mathscr{D}$. The $\mathsf{derived\ category\ of\ a\ DG\text{-}category}$ $\mathscr{D}$ is defined as the Verdier quotient

$$\mathbf{D}(\mathscr{D}) = [\mathsf{dgm}\text{-}\mathscr{D}]/[\mathsf{acycl}\text{-}\mathscr{D}].$$

It has a natural triangulated structure. The quotient functor $[\mathsf{dgm}\text{-}\mathscr{D}] \to \mathbf{D}(\mathscr{D})$ commutes with arbitrary direct sums and its restriction onto the category of perfect DG-modules is fully faithful, $\mathsf{Perf}(\mathscr{D}) \subset \mathbf{D}(\mathscr{D})$. In fact, the category of perfect DG-modules identifies with the subcategory $\mathbf{D}(\mathscr{D})^{\mathsf{comp}}$ of *compact objects* in $\mathbf{D}(\mathscr{D})$ (recall that an object $\mathbf{x}$ of a category is $\mathsf{compact}$ if the functor $\mathsf{Hom}(\mathbf{x}, -)$ commutes with arbitrary direct sums). A DG-category $\mathscr{D}$ is called $\mathsf{pretriangulated}$ if $[\mathscr{D}] \subset \mathbf{D}(\mathscr{D})$ is a triangulated subcategory. If additionally $[\mathscr{D}]$ is closed under direct summands, we will say that $\mathscr{D}$ is $\mathsf{strongly\ pretriangulated}$.

Sometimes it is convenient to have a description of $\mathbf{D}(\mathscr{D})$ not using the Verdier quotient construction. One way is to consider the subcategory of $\mathsf{dgm}\text{-}\mathscr{D}$ consisting of all DG-modules P such that for any acyclic DG-module A the complex $\mathsf{Hom}_{\mathsf{dgm}\text{-}\mathscr{D}}(P, A)$ is acyclic. Such DG-modules P are called $\mathsf{homotopically\ projective}$, or simply $\mathsf{h\text{-}projective}$. The full subcategory of $\mathsf{dgm}\text{-}\mathscr{D}$ consisting of h-projective DG-modules is denoted by $\mathsf{hproj}\text{-}\mathscr{D}$. Its homotopy category is equivalent to the derived category

$$[\mathsf{hproj}\text{-}\mathscr{D}] \cong \mathbf{D}(\mathscr{D}).$$

It is easy to see that any representable DG-module is h-projective.

Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. An $\mathsf{enhancement}$ for $\mathcal{T}$ is a choice of a DG-category $\mathscr{D}$ and of an equivalence $\epsilon : \mathbf{D}(\mathscr{D}) \to \mathcal{T}$ of triangulated categories. An enhancement induces an equivalence $\epsilon : \mathsf{Perf}(\mathscr{D}) \cong \mathcal{T}^{\mathsf{comp}}$.

2.2. **DG-bimodules and tensor products.** If $\mathscr{D}_1$ and $\mathscr{D}_2$ are DG-categories over k, the tensor product $\mathscr{D}_1 \otimes_{\mathsf{k}} \mathscr{D}_2$ is the DG-category whose objects are pairs $(\mathbf{x}_1, \mathbf{x}_2)$ with $\mathbf{x}_i$ being objects of $\mathscr{D}_i$, and with morphisms defined by

$$\mathsf{Hom}_{\mathscr{D}_1 \otimes \mathscr{D}_2}((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) = \mathsf{Hom}_{\mathscr{D}_1}(\mathbf{x}_1, \mathbf{y}_1) \otimes_{\mathsf{k}} \mathsf{Hom}_{\mathscr{D}_2}(\mathbf{x}_2, \mathbf{y}_2).$$

A $\mathscr{D}_1$-$\mathscr{D}_2$ DG-bimodule is a DG-module over $\mathscr{D}_1^{\mathsf{opp}} \otimes \mathscr{D}_2$. In other words, a DG-bimodule φ associates with any pair of objects $\mathbf{x}_1 \in \mathscr{D}_1$, $\mathbf{x}_2 \in \mathscr{D}_2$ a complex $\varphi(\mathbf{x}_1, \mathbf{x}_2)$ and a collection of morphisms of complexes

$$\mathsf{Hom}_{\mathscr{D}_1}(\mathbf{x}_1, \mathbf{y}_1) \otimes \varphi(\mathbf{x}_1, \mathbf{x}_2) \to \varphi(\mathbf{y}_1, \mathbf{x}_2), \qquad \varphi(\mathbf{x}_1, \mathbf{x}_2) \otimes \mathsf{Hom}_{\mathscr{D}_2}(\mathbf{y}_2, \mathbf{x}_2) \to \varphi(\mathbf{x}_1, \mathbf{y}_2)$$

for all $\mathbf{y}_1 \in \mathscr{D}_1$, $\mathbf{y}_2 \in \mathscr{D}_2$, which commute and are compatible with the composition laws in $\mathscr{D}_1$ and $\mathscr{D}_2$. One of the most important examples of a DG-bimodule is the diagonal $\mathscr{D}$-$\mathscr{D}$ DG-bimodule $\mathscr{D}$ defined by

$$\mathscr{D}(\mathbf{x}_1, \mathbf{x}_2) = \mathsf{Hom}_{\mathscr{D}}(\mathbf{x}_2, \mathbf{x}_1).$$

Other examples can be constructed as exterior products of left $\mathscr{D}_1$-modules M with right $\mathscr{D}_2$-modules N

$$(M \otimes_{\mathsf{k}} N)(\mathbf{x}_1, \mathbf{x}_2) = M(\mathbf{x}_1) \otimes_{\mathsf{k}} N(\mathbf{x}_2).$$

Let φ be a $\mathscr{D}_1$-$\mathscr{D}_2$ DG-bimodule and ψ a $\mathscr{D}_2$-$\mathscr{D}_3$ DG-bimodule. Their tensor product over $\mathscr{D}_2$ is a $\mathscr{D}_1$-$\mathscr{D}_3$ DG-bimodule defined by

$$(\varphi \otimes_{\mathscr{D}_2} \psi)(\mathbf{x}_1, \mathbf{x}_3) :=$$
$$\mathsf{Coker}\left(\bigoplus_{\mathbf{x}_2, \mathbf{y}_2 \in \mathscr{D}_2} \varphi(\mathbf{x}_1, \mathbf{x}_2) \otimes_{\mathsf{k}} \mathsf{Hom}_{\mathscr{D}_2}(\mathbf{y}_2, \mathbf{x}_2) \otimes_{\mathsf{k}} \psi(\mathbf{y}_2, \mathbf{x}_3) \to \bigoplus_{\mathbf{x}_2 \in \mathscr{D}_2} \varphi(\mathbf{x}_1, \mathbf{x}_2) \otimes_{\mathsf{k}} \psi(\mathbf{x}_2, \mathbf{x}_3) \right).$$

It is a straightforward verification to see that

$$\varphi \otimes_{\mathscr{D}_2} \mathscr{D}_2 = \varphi \qquad \text{and} \qquad \mathscr{D}_2 \otimes_{\mathscr{D}_2} \psi = \psi.$$

Of course, taking either $\mathscr{D}_1$ or $\mathscr{D}_3$ to be just the base field we obtain the tensor product in the appropriate categories of DG-modules, like $- \otimes_{\mathscr{D}_2} - : \mathsf{dgm}\text{-}\mathscr{D}_2 \times \mathsf{dgm}\text{-}\mathscr{D}_2^{\mathsf{opp}} \to \mathsf{dgm}\text{-}\mathsf{k}$. The same verification as above shows that

$$\varphi \otimes_{\mathscr{D}_2} h_{\mathbf{x}_2} = \varphi(-, \mathbf{x}_2) \qquad \text{and} \qquad h^{\mathbf{x}_1} \otimes_{\mathscr{D}_1} \phi = \phi(\mathbf{x}_1, -)$$

for any $\mathscr{D}_1$-$\mathscr{D}_2$-bimodule ϕ and any objects $\mathbf{x}_1 \in \mathscr{D}_1$, $\mathbf{x}_2 \in \mathscr{D}_2$.

The derived tensor product is defined by replacing etiher of the factors by an h-projective resolution

$$\varphi \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_2} \psi := P(\varphi) \otimes_{\mathscr{D}_2} \psi \cong \varphi \otimes_{\mathscr{D}_2} P(\psi).$$

It is a bifunctor on derived categories $\mathbf{D}(\mathscr{D}_1^{\mathsf{opp}} \otimes \mathscr{D}_2) \times \mathbf{D}(\mathscr{D}_2^{\mathsf{opp}} \otimes \mathscr{D}_3) \to \mathbf{D}(\mathscr{D}_1^{\mathsf{opp}} \otimes \mathscr{D}_3)$. Since representable modules are h-projective, we have

$$\varphi \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_2} h_{\mathbf{x}_2} = \varphi(-, \mathbf{x}_2) \qquad \text{and} \qquad h^{\mathbf{x}_1} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_1} \phi = \phi(\mathbf{x}_1, -)$$

Also, there is a nice choice of an h-projective resolution for the diagonal bimodule, called the bar-resolution. It is defined by

$$\mathbf{B}(\mathscr{D}) = \bigoplus_{p=0}^{\infty} \bigoplus_{\mathbf{x}_0, \dots, \mathbf{x}_p \in \mathscr{D}} h_{\mathbf{x}_p} \otimes_{\mathsf{k}} \mathscr{D}(\mathbf{x}_p, \mathbf{x}_{p-1}) \otimes_{\mathsf{k}} \dots \otimes_{\mathsf{k}} \mathscr{D}(\mathbf{x}_1, \mathbf{x}_0) \otimes_{\mathsf{k}} h^{\mathbf{x}_0}[p] \tag{1}$$

with the differential consisting of the differentials of $h_{\mathbf{x}_p}$, $h^{\mathbf{x}_0}$, and $\mathscr{D}(\mathbf{x}_i, \mathbf{x}_{i-1})$ and of the compositions $h_{\mathbf{x}_p} \otimes \mathscr{D}(\mathbf{x}_p, \mathbf{x}_{p-1}) \to h_{\mathbf{x}_{p-1}}$, $\mathscr{D}(\mathbf{x}_{i+1}, \mathbf{x}_i) \otimes \mathscr{D}(\mathbf{x}_i, \mathbf{x}_{i-1}) \to \mathscr{D}(\mathbf{x}_{i+1}, \mathbf{x}_{i-1})$ and $\mathscr{D}(\mathbf{x}_1, \mathbf{x}_0) \otimes h^{\mathbf{x}_0} \to h^{\mathbf{x}_1}$. Using this resolution it is easy to see that

$$\varphi \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_2} \mathscr{D}_2 \cong \varphi \qquad \text{and} \qquad \mathscr{D}_2 \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_2} \psi \cong \psi.$$

2.3. **Enhancements for derived categories of quasicoherent sheaves.** There are several ways to construct an enhancement for the derived category of quasicoherent sheaves on a given scheme X. It turn out that a bit easier is to use a cosimplicial enhancement and then induce a DG-enhancement from that. We refer to the Appendix of [GM] for more details on normalization complexes and Alexander–Whitney and Eilenberg–Zilber maps for comsimplicial objects. Here we just give the required definitions.

For any category $\mathscr{A}$ a cosimplicial object in $\mathscr{A}$ is just a functor $\Delta \to \mathscr{A}$, where Δ is the category of finite nonempty linearly ordered sets. We denote by $[p] := \{0, 1, \dots, p\} \in \Delta$, $p \geq 0$, (with the natural order) the objects of Δ. Then a cosimplicial object $A : \Delta^{\mathsf{opp}} \to \mathscr{A}$ is given by a collection of objects $A^p := A([p]) \in \mathscr{A}$ and a collection of morphisms $f^* : A^p \to A^q$ for any nondecreasing map $f : [p] \to [q]$. The unique strictly increasing map $[p-1] \to [p]$ with the set of values $[p] \setminus \{i\}$ (as well as the induced morphism A_p) is called the i-th face map and is denoted by ∂_p^i. Analogously, the unique surjective nondecreasing map $[p+1] \to [p]$ which takes value i twice is called the i-th degeneration map and is denoted by s_p^i.

Assume $\mathscr{A}$ is abelian. Given a cosimplicial object $A^\bullet$ in $\mathscr{A}$ one defines the corresponding normalized complex by

$$(NA)^p = \bigcap_{i=0}^{p-1} \mathsf{Ker}\left(A^p \xrightarrow{s_{p-1}^i} A^{p-1}\right), \qquad d_{NA}^p = \sum_{i=0}^{p+1} (-1)^i \partial_{p+1}^i.$$

Assume moreover that $\mathscr{A}$ is a monoidal category. Then the functor N is a lax monoidal functor. It means that for any cosimplicial objects $A^\bullet, B^\bullet$ in $\mathscr{A}$ there is a morphism of functors

$$\Delta_{A,B} : N(A) \otimes N(B) \to N(A \otimes B)$$

called the Alexander–Whitney map and defined by

$$\Delta_{A,B}(a \otimes b) = (\partial_{p+q}^{p+q} \circ \dots \circ \partial_{p+1}^{p+1})(a) \otimes (\partial_{p+q}^0 \circ \dots \circ \partial_{q+1}^0)(b), \qquad a \in A^p,\ b \in B^q,$$

as well as a morphism of functors

$$\nabla_{A,B} : N(A \otimes B) \to N(A) \otimes N(B),$$

called the Eilenberg–Zilber map and defined by

$$\nabla_{A,B}(a \otimes b) = \sum_{p+q=n} \sum_{\tau \in \mathsf{Shuf}(p,q)} (-1)^{|\tau|} (s_p^{\tau_{p+1}-1} \circ \dots \circ s_{p+q-1}^{\tau_{p+q}-1})(a) \otimes (s_1^{\tau_1 - 1} \circ \dots \circ s_{p+q-1}^{\tau_p - 1})(b), \quad a \in A^n,\ b \in B^n.$$

Here $\mathsf{Shuf}(p,q)$ is the set of all $(p+q)$ permutations τ such that $\tau_1 < \dots < \tau_p$ and $\tau_{p+1} < \dots < \tau_{p+q}$ (such permutations are called (p,q)-shuffles), and $|\tau|$ is the parity of the permutation τ. One can check that

$$\nabla_{A,B} \circ \Delta_{A,B} = \mathsf{id}, \qquad \Delta_{A,B} \circ \nabla_{A,B} \sim \mathsf{id},$$

where $\sim$ means homotopic. In particular, both maps are quasiisomorphisms.

One of the ways to construct an enhancement is the following. First we construct a cosimplicial enhancement. Assume for simplicity that X is smooth and projective and choose a finite affine covering $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ for X. Let $\mathscr{C} = \mathscr{C}_X$ be the cosimplicial category with objects being finite complexes of vector bundles on X and with $\mathsf{Hom}_{\mathscr{C}}(F, G)$ being the cosimplicial complex of the Čech chains of the complex $F^\vee \otimes G$. Explicitly

$$\mathsf{Hom}_{\mathscr{C}}(F, G)_p = \bigoplus_{i,j \in \mathbb{Z}} \bigoplus_{(\alpha_0, \dots, \alpha_p) \in A^{p+1}} \Gamma(\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_p}, (F^i)^\vee \otimes G^j)[i-j] \tag{2}$$

(we allow repetitions of indices α_s) with the differential induced by the differentials in the complexes F and G and with evident face and degeneration maps. Then it is easy to see that $\mathscr{C}_X$ is a cosimplicial category.

Further we define a DG-category $\mathscr{D} = \mathscr{D}_X$ by taking the same objects as in $\mathscr{C}_X$ and with Hom-complexes being normalized complexes of cosimplicial Hom-sets in $\mathscr{C}_X$. In other words

$$\mathsf{Hom}_{\mathscr{D}}(F,G) = N\,\mathsf{Hom}_{\mathscr{C}}(F,G). \tag{3}$$

To define the composition law we use the Alexander–Whitney map and the composition law in $\mathscr{C}_X$:

$$\mathsf{Hom}_{\mathscr{D}}(F,G)\otimes\mathsf{Hom}_{\mathscr{D}}(G,H) = N\,\mathsf{Hom}_{\mathscr{C}}(F,G)\otimes N\,\mathsf{Hom}_{\mathscr{C}}(G,H) \xrightarrow{\Delta}$$
$$\xrightarrow{\Delta} N(\mathsf{Hom}_{\mathscr{C}}(F,G)\otimes\mathsf{Hom}_{\mathscr{C}}(G,H)) \xrightarrow{N(m_{\mathscr{C}})} N\,\mathsf{Hom}_{\mathscr{C}}(F,G) = \mathsf{Hom}_{\mathscr{D}}(F,G).$$

It is associative because both the Alexander–Whitney map and the composition law in $\mathscr{C}$ are (and by functoriality of Δ).

It is clear that $[\mathscr{D}_X] \cong \mathbf{D}^{\mathsf{perf}}(X)$. Indeed, the categories have the same objects and the spaces of morphisms are also the same (because we can compute the cohomology by Čech complex). Moreover, it is easy to see that Then $\mathbf{D}(\mathscr{D}) \cong \mathbf{D}(X)$. Indeed, it is easy to construct a functor $\mathbf{D}(X) \to \mathbf{D}(\mathscr{D})$ which takes an object $G \in \mathbf{D}(X)$ to the DG-module M_G such that $M_G(F)$ is the RHS of (3) and (2) for any $F \in \mathscr{D}$. The same argument shows that this functor is an equivalence. Thus the DG-category $\mathscr{D}$ provides an enhancement of $\mathbf{D}(X)$. We will refer to this enhancement as the Čech enhancement with respect to the covering $\{\mathcal{U}_\alpha\}$. Note that $[\mathscr{D}] \cong \mathbf{D}^b(\mathsf{coh}(X))$, so the Čech enhancement is strongly pretriangulated.

One nice property of the Čech enhancement is that

$$\mathsf{Hom}_{\mathscr{D}}(F^\vee, G^\vee) = \mathsf{Hom}_{\mathscr{D}}(G,F).$$

This means that we have in fact two enhancements

$$\epsilon : \mathbf{D}(\mathscr{D}) \xrightarrow{\cong} \mathbf{D}(X) \qquad \text{and} \qquad \phi : \mathbf{D}(\mathscr{D}^{\mathsf{opp}}) \xrightarrow{\cong} \mathbf{D}(X) \qquad \text{with} \qquad \phi(\mathbf{x}) = \epsilon(\mathbf{x})^\vee.$$

In particular, it follows that for each left $\mathscr{D}$-module M and right $\mathscr{D}$-module N we have

$$M \overset{\mathbb{L}}{\otimes}_{\mathscr{D}} N \cong \mathbf{H}^\bullet(X, \epsilon(M)^\vee \overset{\mathbb{L}}{\otimes} \epsilon(N)). \tag{4}$$

Indeed, for representable DG-modules $M = h_{\mathbf{x}}$ and $N = h^{\mathbf{y}}$ the LHS gives $\mathbf{H}^\bullet(\mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{y}))$ which is nothing but $\mathsf{Ext}^\bullet(\epsilon(\mathbf{x}),\epsilon(\mathbf{y})) \cong \mathbf{H}^\bullet(X, \epsilon(\mathbf{x})^\vee \overset{\mathbb{L}}{\otimes} \epsilon(\mathbf{y}))$ which coincides with the RHS. For arbitrary DG-modules the isomorphism follows by devissage.

Now consider the square $X \times X$, its open affine covering by $\{\mathcal{U}_\alpha \times \mathcal{U}_\beta\}$, and the corresponding cosimplicial and DG Čech enhancements $\mathscr{C}_{X\times X}$ and $\mathscr{D}_{X\times X}$ respectively. It is easy to see that for any $F_1, F_2, G_1, G_2 \in \mathbf{D}^b(\mathsf{coh}(X))$ we have

$$\begin{aligned}
&\mathsf{Hom}_{\mathscr{C}_{X\times X}}(F_2 \boxtimes F_1^\vee, G_2 \boxtimes G_1^\vee) = \\
&\quad = \bigoplus \Gamma((\mathcal{U}_{\alpha_0}\times\mathcal{U}_{\beta_0})\cap\dots\cap(\mathcal{U}_{\alpha_p}\times\mathcal{U}_{\beta_p}), (F_2^{i_2}\boxtimes(F_1^{i_1})^\vee)^\vee \otimes (G_2^{j_2}\boxtimes(G_1^{j_1})^\vee)) = \\
&\quad = \bigoplus \Gamma(\mathcal{U}_{\beta_0}\cap\dots\cap\mathcal{U}_{\beta_p}, F_1^{i_1}\otimes(G_1^{j_1})^\vee)\otimes\Gamma(\mathcal{U}_{\alpha_0}\cap\dots\cap\mathcal{U}_{\alpha_p}, (F_2^{i_2})^\vee\otimes G_2^{j_2}) = \\
&\quad = \mathsf{Hom}_{\mathscr{C}^{\mathsf{opp}}}(F_1,G_1)\otimes\mathsf{Hom}_{\mathscr{C}}(F_2,G_2).
\end{aligned}$$

Moreover, this isomorphism is compatible with the composition laws. In other words, it gives a cosimplicial functor

$$\mathscr{C}_X^{\mathsf{opp}} \otimes \mathscr{C}_X \to \mathscr{C}_{X\times X}, \qquad (F_1,F_2) \mapsto F_2 \boxtimes F_1^\vee$$

which is fully faithful. We combine this functor with the Alexander–Whitney map to construct a functor of DG-enhancements

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}^{\mathsf{opp}}}(F_1,G_1)\otimes\mathsf{Hom}_{\mathscr{D}}(F_2,G_2) = N\,\mathsf{Hom}_{\mathscr{C}^{\mathsf{opp}}}(F_1,G_1)\otimes N\,\mathsf{Hom}_{\mathscr{C}}(F_2,G_2) \xrightarrow{\Delta} \\ \xrightarrow{\Delta} N(\mathsf{Hom}_{\mathscr{C}^{\mathsf{opp}}}(F_1,G_1)\otimes\mathsf{Hom}_{\mathscr{C}}(F_2,G_2)) \cong \\ \cong N\,\mathsf{Hom}_{\mathscr{C}_{X\times X}}(F_2\boxtimes F_1^\vee, G_2\boxtimes G_1^\vee) = \mathsf{Hom}_{\mathscr{D}_{X\times X}}(F_2\boxtimes F_1^\vee, G_2\boxtimes G_1^\vee).\end{aligned}$$

Since the Alexander–Whitney map is a quasiisomorphism, the constructed DG-functor is quasi fully faithful. It follows that it induces equivalences

$$\mathbf{D}(\mathscr{D}_X^{\mathsf{opp}}\otimes\mathscr{D}_X)\cong\mathbf{D}(\mathscr{D}_{X\times X})\cong\mathbf{D}(X\times X)\qquad\text{and}\qquad \mathsf{Perf}(\mathscr{D}_X^{\mathsf{opp}}\otimes\mathscr{D}_X)\cong[\mathscr{D}_{X\times X}]\cong\mathbf{D}^b(\mathsf{coh}(X\times X)).$$

We denote these equivalences by μ. By definition of μ we have

$$\mu(h_{\mathbf{x}}\otimes_{\mathsf{k}} h^{\mathbf{y}}) = \epsilon(\mathbf{y})\boxtimes\epsilon(\mathbf{x})^\vee.$$

Finally, denoting by $K\mapsto K^T$ the transposition of $K\in\mathbf{D}(X\times X)$ (the pullback under the permutation of factors involution of $X\times X$) we have

$$\kappa\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}}\kappa' \cong \mathbf{H}^\bullet(X\times X, \mu(\kappa)\overset{\mathbb{L}}{\otimes}\mu(\kappa')^T) \tag{5}$$

for all $\mathscr{D}$-$\mathscr{D}$-bimodules κ and κ'. Indeed, if $\kappa = h_{\mathbf{x}}\otimes h^{\mathbf{y}}$ and $\kappa' = h_{\mathbf{x}'}\otimes h^{\mathbf{y}'}$ then the formula is correct

$$\begin{aligned}\mathbf{H}^\bullet(X\times X,\mu(\kappa)\overset{\mathbb{L}}{\otimes}\mu(\kappa')^T)\cong\mathbf{H}^\bullet(X\times X,(\epsilon(\mathbf{y})\boxtimes\epsilon(\mathbf{x})^\vee)\overset{\mathbb{L}}{\otimes}(\epsilon(\mathbf{x}')^\vee\boxtimes\epsilon(\mathbf{y}')))\cong \\ \mathbf{H}^\bullet(X,\epsilon(\mathbf{y})\overset{\mathbb{L}}{\otimes}\epsilon(\mathbf{x}')^\vee)\otimes\mathbf{H}^\bullet(X,\epsilon(\mathbf{x})^\vee\overset{\mathbb{L}}{\otimes}\epsilon(\mathbf{y}'))\cong(h_{\mathbf{x}'}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}}h^{\mathbf{y}})\otimes_{\mathsf{k}}(h_{\mathbf{x}}\overset{\mathbb{L}}{\otimes}_{\mathscr{D}}h^{\mathbf{y}'})\cong\kappa\overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}}\kappa'\end{aligned}$$

(we use (4) in the third isomorphism). For arbitrary DG-bimodules the statement follows by devissage.

Let us look at the image of the diagonal bimodule under this equivalence. Let $\Delta : X\to X\times X$ be the diagonal embedding.

Lemma 2.1. *We have* $\mu(\mathscr{D})\cong\Delta_*\mathcal{O}_X$.

Proof. By definition of μ the bimodule corresponding to $\Delta_*\mathcal{O}_X$ evaluated on objects $\mathbf{x},\mathbf{y}\in\mathscr{D}$ is

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}_{X\times X}}(\mu(h_{\mathbf{x}}\otimes h^{\mathbf{y}}),\Delta_*\mathcal{O}_X) = \mathsf{Hom}_{\mathscr{D}_{X\times X}}(\epsilon(\mathbf{y})\boxtimes\epsilon(\mathbf{x})^\vee,\Delta_*\mathcal{O}_X) = \\ = \mathbf{H}^\bullet(X\times X,(\epsilon(\mathbf{y})^\vee\boxtimes\epsilon(\mathbf{x}))\overset{\mathbb{L}}{\otimes}\Delta_*\mathcal{O}_X)\cong\mathbf{H}^\bullet(X,\mathbb{L}\Delta^*(\epsilon(\mathbf{y})^\vee\boxtimes\epsilon(\mathbf{x})))\cong \\ \cong\mathbf{H}^\bullet(X,\epsilon(\mathbf{y})^\vee\overset{\mathbb{L}}{\otimes}\epsilon(\mathbf{x}))\cong\mathsf{Hom}_{\mathscr{D}}(\mathbf{y},\mathbf{x})\end{aligned}$$

(here $\mathbf{H}^\bullet$ stands for the Čech complex), and the RHS is precisely $\mathscr{D}(\mathbf{x},\mathbf{y})$. □

On the other hand, consider the $\mathscr{D}$-bimodule $\mathscr{D}^\vee$ corresponding to the inverse Serre fuctor of $\mathbf{D}(X)$

$$S^{-1}(F) := F\otimes\omega_X^{-1}[-\dim X]. \tag{6}$$

In other words, we define

$$\mathscr{D}^\vee(\mathbf{x},\mathbf{y}) = \mathsf{Hom}_{\mathscr{D}}(\epsilon(\mathbf{y}),S^{-1}(\epsilon(\mathbf{x}))). \tag{7}$$

Lemma 2.2. *We have* $\mu(\mathscr{D}^\vee)\cong\Delta_*\omega_X^{-1}[-\dim X]$.

Proof. By definition of μ the bimodule corresponding to $\Delta_*\omega_X^{-1}[-\dim X]$ evaluated on objects $\mathbf{x}, \mathbf{y} \in \mathscr{D}$ is

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}_{X\times X}}(\mu(h_{\mathbf{x}} \otimes h^{\mathbf{y}}), \Delta_*\omega_X^{-1}[-\dim X]) = \mathsf{Hom}_{\mathscr{D}_{X\times X}}(\epsilon(\mathbf{y}) \boxtimes \epsilon(\mathbf{x})^\vee, \Delta_*\omega_X^{-1}[-\dim X]) = \\ = \mathbf{H}^\bullet(X \times X, (\epsilon(\mathbf{y})^\vee \boxtimes \epsilon(\mathbf{x})) \overset{\mathbb{L}}{\otimes} \Delta_*\omega_X^{-1}[-\dim X]) \cong \mathbf{H}^\bullet(X, \mathbb{L}\Delta^*(\epsilon(\mathbf{y})^\vee \boxtimes \epsilon(\mathbf{x})) \otimes \omega_X^{-1}[-\dim X]) \cong \\ \cong \mathbf{H}^\bullet(X, \epsilon(\mathbf{y})^\vee \overset{\mathbb{L}}{\otimes} \epsilon(\mathbf{x}) \otimes \omega_X^{-1}[-\dim X]) \cong \mathsf{Hom}_{\mathscr{D}}(\mathbf{y}, S^{-1}(\mathbf{x}))\end{aligned}$$

(here again $\mathbf{H}^\bullet$ stands for the Čech complex), and the RHS is precisely $\mathscr{D}^\vee(\mathbf{x}, \mathbf{y})$. □

Remark 2.3. In fact it is easy to check that

$$\mathscr{D}^\vee \cong \mathsf{RHom}_{\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})}(\mathscr{D}, \mathscr{D} \otimes_{\mathsf{k}} \mathscr{D}),$$

which explains the notation.

2.4. **Semiorthogonal decompositions.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ full triangulated subcategories. One says that

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$$

is a semiorthogonal decomposition if

- for all $A \in \mathcal{A}$, $B \in B$ one has $\mathsf{Hom}(B, A) = 0$, and
- for any $T \in \mathcal{T}$ there is a distinguished triangle $T_{\mathcal{B}} \to T \to T_{\mathcal{A}}$ with $T_{\mathcal{A}} \in \mathcal{A}$ and $T_{\mathcal{B}} \in \mathcal{B}$.

Note that the above triangle is unique because of the semiorthogonality.

Now assume that we are given a semiorthogonal decomposition

$$\mathbf{D}^b(\mathsf{coh}(X)) = \langle \mathcal{A}, \mathcal{B} \rangle$$

with X smooth and projective. It was shown in [K1] that there is a semiorthogonal decomposition

$$\mathbf{D}^b(\mathsf{coh}(X \times X)) = \langle \mathcal{A}_X, \mathcal{B}_X \rangle,$$

where the subcategories $\mathcal{A}_X$ and $\mathcal{B}_X$ are the minimal closed under direct summands triangulated subcategories of $\mathbf{D}^b(\mathsf{coh}(X \times X))$ containing all objects of the form $A \boxtimes F$ (resp. $B \boxtimes F$) with arbitrary $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $F \in \mathbf{D}^b(\mathsf{coh}(X))$. Consequently, we can consider the induced decomposition of the structure sheaf of the diagonal

$$Q \to \Delta_*\mathcal{O}_X \to P$$

with $P \in \mathcal{A}_X$, $Q \in \mathcal{B}_X$. One can easily see that the Fourier–Mukai functors $\mathbf{D}^b(\mathsf{coh}(X)) \to \mathbf{D}^b(\mathsf{coh}(X))$ associated with the kernels P and Q take any object $F \in \mathbf{D}^b(\mathsf{coh}(X))$ to its components $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ with respect to the initial semiorthogonal decomposition. Thus the above triangle can be considered as the universal semiorthogonal decomposition triangle.

It turns out that the kernel Q has a nice interpretation in terms of the natural enhancements of the DG-categories $\mathbf{D}^b(\mathsf{coh}(X))$ and $\mathcal{B}$. This interpretation is crucial for the rest of the paper.

Consider the Čech enhancement $\mathscr{D}$ of $\mathbf{D}^b(\mathsf{coh}(X))$. Let $\mathscr{A} \subset \mathscr{D}$ and $\mathscr{B} \subset \mathscr{D}$ be the full DG-subcategories consisting of all objects contained in $\mathcal{A} \subset [\mathscr{D}]$ and $\mathcal{B} \subset [\mathscr{D}]$ respectively. Consider the tensor product $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}$. Here the first factor is considered as a $\mathscr{D}$-$\mathscr{B}$-bimodule and the second factor is considered as a $\mathscr{B}$-$\mathscr{D}$-bimodule by restricting the corresponding arguments to $\mathscr{B}$. Let $\mu : \mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D}) \cong \mathbf{D}(X \times X)$ be the induced enhancement.

Proposition 2.4. *We have $Q \cong \mu(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D})$.*

Proof. Denote $Q' := \mu(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}) \in \mathbf{D}(X \times X)$. The composition law in $\mathscr{D}$ induces a morphism of $\mathscr{D}$-$\mathscr{D}$-bimodules $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D} \to \mathscr{D}$ which under the functor μ gives a morphism $Q' \to \Delta_*\mathcal{O}_X$ in $\mathbf{D}(X \times X)$. We denote by P' its cone, so that we have a distinguished triangle

$$Q' \to \Delta_*\mathcal{O}_X \to P'$$

in $\mathbf{D}(X \times X)$. We need to show that $Q \cong Q'$. By definition of Q and P it is enough for this to check that $P' \in \mathcal{A}_X$ and $Q' \in \mathcal{B}_X$.

Note that $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}$ can be rewritten as $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}$. To compute this derived tensor product we can use the bar-resolution of $\mathscr{B}$. We deduce that

$$\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D} \cong \bigoplus_{p=0}^{\infty} \bigoplus_{\mathbf{x}_0,\dots,\mathbf{x}_p \in \mathscr{B}} h_{\mathbf{x}_p} \otimes \mathscr{B}(\mathbf{x}_p, \mathbf{x}_{p-1}) \otimes \dots \otimes \mathscr{B}(\mathbf{x}_1, \mathbf{x}_0) \otimes h^{\mathbf{x}_0}[p] \tag{8}$$

where the factors $h_{\mathbf{x}_p}$ and $h^{\mathbf{x}_0}$ are considered as $\mathscr{D}$-modules. Applying μ we deduce that $Q' = \mu(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D})$ is contained in the subcategory generated by $\epsilon(\mathbf{x}_0) \boxtimes \epsilon(\mathbf{x}_p)^\vee$ with $\mathbf{x}_0, \mathbf{x}_p \in \mathscr{B}$. In particular, $Q' \in \mathcal{B}_X$.

On the other hand, for any $\mathbf{b} \in \mathscr{B}$ and $\mathbf{y} \in \mathscr{D}$ we have

$$\begin{aligned}\mathsf{Hom}_{\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})}(h_{\mathbf{y}} \otimes h^{\mathbf{b}}, \mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}) \cong \mathscr{D}(\mathbf{y}, -) \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}(-, \mathbf{b}) \cong \\ \cong \mathscr{D}(\mathbf{y}, -) \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} h_{\mathbf{b}} \cong \mathscr{D}(\mathbf{y}, \mathbf{b}) \cong \mathsf{Hom}_{\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})}(h_{\mathbf{y}} \otimes h^{\mathbf{b}}, \mathscr{D})\end{aligned}$$

and the isomorphism is induced by the morphism $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D} \to \mathscr{D}$. Therefore the cone of that morphism is orthogonal to all bimodules of the form $h_{\mathbf{y}} \otimes h^{\mathbf{b}}$ in $\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})$. Applying the functor μ we conclude that the the object P' is orthogonal to all objects $\epsilon(\mathbf{b}) \boxtimes \epsilon(\mathbf{y})^\vee$ in $\mathbf{D}(X \times X)$. The latter generate the subcategory $\mathcal{B}_X$. Thus P' is in the right orthogonal to $\mathcal{B}_X$, hence $P' \in \mathcal{A}_X$. $\square$

3. Normal Hocschild cohomology

3.1. Hochschild cohomology. The Hoschschild cohomology of a DG-category is defined as

$$\mathsf{HH}^\bullet(\mathscr{D}) = \mathsf{Ext}^\bullet_{\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})}(\mathscr{D}, \mathscr{D}).$$

For an enhanced triangulated category the Hochschild cohomology is defined as the Hochschild cohomology of the enhancement. The Hochschild cohomology of $\mathbf{D}(X)$ will be denoted by $\mathsf{HH}^\bullet(X)$.

Note that using the equivalence μ one can identify

$$\mathsf{HH}^\bullet(X) \cong \mathsf{Ext}^\bullet(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \cong \mathbf{H}^\bullet(X, \Delta^!\Delta_*\mathcal{O}_X). \tag{9}$$

where $\Delta^! : \mathbf{D}(X \times X) \to \mathbf{D}(X)$

$$\Delta^!(F) = \mathbb{L}\Delta^*(F) \otimes \omega_X^{-1}[-\dim X] \tag{10}$$

is the right adjoint functor of $\Delta_* : \mathbf{D}(X) \to \mathbf{D}(X \times X)$.

Computing the RHS of (9) one obtains the Hochschild–Kostant–Rosenberg isomorphism

$$\mathsf{HH}^t(X) = \bigoplus_{p+q=t} H^q(X, \Lambda^p T_X).$$

In particular, the Hochschild cohomology of X lives in nonnegative degrees and $\mathsf{HH}^0(X) = \mathsf{k}$ if X is connected.

Now assume that a semiorthogonal decomposition

$$\mathbf{D}^b(\mathsf{coh}(X)) = \langle \mathcal{A}, \mathcal{B} \rangle$$

is given. The Čech enhancement of $\mathbf{D}(X)$ induces natural enhancements of $\mathcal{A}$ and $\mathcal{B}$, so we can speak about Hochschild cohomology of these categories. Recall the induced semiorthogonal decomposition

$$\mathbf{D}^b(\mathsf{coh}(X \times X)) = \langle \mathcal{A}_X, \mathcal{B}_X \rangle,$$

and the distinguished triangle

$$Q \to \Delta_* \mathcal{O}_X \to P \tag{11}$$

with $Q \in \mathcal{B}_X$ and $P \in \mathcal{A}_X$. Furthermore, as it was shown in [K2] there is an isomorphism

$$\mathsf{HH}^\bullet(\mathcal{A}) \cong \mathbf{H}^\bullet(X, \Delta^! P),$$

analogous to (9), and the restriction morphism $\mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A})$ of Hochschild cohomology is induced by the morphism $\Delta_* \mathcal{O}_X \to P$ from (11). Consequently, one has a distinguished triangle

$$\mathbf{H}^\bullet(X, \Delta^! Q) \to \mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A}). \tag{12}$$

The first term of this triangle can be thought of as a complex controlling the restriction map of Hochschild cohomology. Our goal is to show how it can be computed in terms of the category $\mathcal{B}$, especially in the case when $\mathcal{B}$ is generated by an exceptional collection.

Lemma 3.1. *We have* $\mathbf{H}^\bullet(X, \Delta^! Q) \cong \mathbf{H}^\bullet(X \times X, Q \otimes \Delta_* \omega_X^{-1}[-\dim X])$.

Proof. This evidently follows from (10). Indeed, we have

$$\mathbf{H}^\bullet(X, \Delta^! Q) \cong \mathbf{H}^\bullet(X, \Delta^* Q \otimes \omega_X^{-1}[-\dim X]) \cong$$
$$\cong \mathbf{H}^\bullet(X \times X, \Delta_*(\Delta^* Q \otimes \omega_X^{-1}[-\dim X])) \cong \mathbf{H}^\bullet(X \times X, Q \overset{\mathbb{L}}{\otimes} \Delta_* \omega_X^{-1}[-\dim X])$$

(the last isomorphism is the projection formula). □

3.2. **The normal bimodule.** Let $\mathscr{D}$ be a DG-category and $\mathscr{B} \subset \mathscr{D}$ a DG-subcategory.

Definition 3.2. The $\mathsf{normal\ bimodule}$ of the embedding $\mathscr{B} \to \mathscr{D}$ is the restriction $\mathscr{D}^\vee_{\mathscr{B}}$ to $\mathscr{B}$ of the $\mathscr{D}$-$\mathscr{D}$-bimodule $\mathscr{D}^\vee$. The $\mathsf{normal\ Hochschild\ cohomology}$ of $\mathscr{B}$ in $\mathscr{D}$ is defined as the derived tensor product of the diagonal and the normal bimodule of $\mathscr{B}$

$$\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D}) := \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}^{\mathsf{opp}} \otimes \mathscr{B}} \mathscr{D}^\vee_{\mathscr{B}}.$$

In case when $\mathscr{D}$ is the Čech enhancement of $\mathbf{D}(X)$ and $\mathscr{B}$ is the induced enhancement of a subcategory $\mathcal{B} \subset \mathbf{D}^b(\mathsf{coh}(X))$ we will wirite $\mathsf{NHH}^\bullet(\mathcal{B}, X)$ instead of $\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D})$.

Let $\mathscr{D}$ be the Čech enhancement of $\mathbf{D}^b(\mathsf{coh}(X))$ discussed above and let $\mu : \mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D}) \to \mathbf{D}(X \times X)$ be the induced equivalence.

Theorem 3.3. *If* $\mathbf{D}^b(\mathsf{coh}(X)) = \langle \mathcal{A}, \mathcal{B} \rangle$ *is a semiorthogonal decomposition then there is a distinguished triangle*

$$\mathsf{NHH}^\bullet(\mathcal{B}, X) \to \mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A}). \tag{13}$$

Proof. By Lemma 3.1 it is enough to identify $\mathbf{H}^\bullet(X \times X, Q \overset{\mathbb{L}}{\otimes} \Delta_* \omega_X^{-1}[-\dim X])$ with the normal Hochschild cohomology. Recall that by Proposition 2.4 we have an isomorphism $Q \cong \mu(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D})$ and by Lemma 2.2 an isomorphism $\Delta_* \omega_X^{-1}[-\dim X] \cong \mu(\mathscr{D}^\vee)^T$. Therefore by (5) we have

$$\mathbf{H}^\bullet(X \times X, Q \overset{\mathbb{L}}{\otimes} \Delta_* \omega_X^{-1}[-\dim X]) \cong (\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D}} \mathscr{D}^\vee.$$

On the other hand,

$$\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D} \cong \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}^{\mathsf{opp}} \otimes \mathscr{B}} (\mathscr{D} \otimes_{\mathsf{k}} \mathscr{D}),$$

hence

$$(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee \cong \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}^{\mathsf{opp}}\otimes\mathscr{B}} (\mathscr{D} \otimes_{\mathsf{k}} \mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee \cong \mathscr{B} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}^{\mathsf{opp}}\otimes\mathscr{B}} \mathscr{D}^\vee_{\mathscr{B}}$$

and this is precisely the normal Hochschild cohomology. □

Remark 3.4. Analogous result can be proved for arbitrary smooth strongly pretriangulated DG-category $\mathscr{D}$. If a semiorthognal decomposition $[\mathscr{D}] = \langle \mathcal{A}, \mathcal{B} \rangle$ is given and $\mathscr{A}, \mathscr{B} \subset \mathscr{D}$ are the DG-subcategories underlying $\mathcal{A}$ and $\mathcal{B}$ respectively, then there is a distinguished triangle

$$\mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D}) \to \mathsf{HH}^\bullet(\mathscr{D}) \to \mathsf{HH}^\bullet(\mathscr{A}).$$

The proof is completely analogous to the one described here. One considers a distinguished triangle $\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D} \to \mathscr{D} \to P'$ in $\mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})$ and verifies that $\mathsf{HH}^\bullet(\mathscr{D}) = \mathsf{Ext}^\bullet(\mathscr{D}, \mathscr{D}) \cong \mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee$ (here the smoothness of $\mathscr{D}$ is important) and that $\mathsf{HH}^\bullet(\mathscr{A}) \cong \mathsf{Ext}^\bullet(P', P') \cong \mathsf{Ext}^\bullet(\mathscr{D}, P') \cong P' \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee$ (the first isomorphism here is the most subtle part; to prove it one constructs a fully faithful embedding $\mathbf{D}(\mathscr{A}^{\mathsf{opp}} \otimes \mathscr{A}) \to \mathbf{D}(\mathscr{D}^{\mathsf{opp}} \otimes \mathscr{D})$ which takes the diagonal $\mathscr{A}$ to P', this embedding is not the trivial one, in fact it is the embedding induced by the embedding $\mathscr{A} \to \mathscr{D}$ and the "mutated" embedding $\mathcal{A} \cong {}^\perp\mathcal{B} \subset [\mathscr{D}]$). This allows to identify the cone of the morphism $\mathsf{HH}^\bullet(\mathscr{D}) \to \mathsf{HH}^\bullet(\mathscr{A})$ with the (shift of) $(\mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{B}} \mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee$ which is just the normal Hochschild cohomology.

3.3. **The normal Hochschild cohomology of an exceptional collection.** Consider the case when the subcategory $\mathcal{B} \subset \mathbf{D}^b(\mathsf{coh}(X))$ is generated by an exceptional collection,

$$\mathcal{B} = \langle E_1, \dots, E_n \rangle.$$

As in the previous section we consider the Čech enhancement $\mathscr{D}$ of $\mathbf{D}^b(\mathsf{coh}(X))$ and take $\mathscr{E}$ to be its DG-subcategory with n objects — $E_1, \dots, E_n$. Note that $\mathscr{E}$ is a subcategory of the DG-category $\mathscr{B}$ considered above.

By definition of the normal bimodule of $\mathscr{E}$ in $\mathscr{D}$ we have

$$\mathscr{D}^\vee_{\mathscr{E}}(E_i, E_j) = \mathsf{Hom}_{\mathscr{D}}(E_j, S^{-1}(E_i)). \tag{14}$$

Lemma 3.5. *The normal cohomology of $\mathscr{E}$ and $\mathscr{B}$ in $\mathscr{D}$ are the same, i.e.* $\mathsf{NHH}^\bullet(\mathscr{E}, \mathscr{D}) \cong \mathsf{NHH}^\bullet(\mathscr{B}, \mathscr{D})$.

Proof. Follows from the fact that the restriction of DG-bimodules from $\mathscr{B}$ to $\mathscr{E}$ induces an equivalence $\mathbf{D}(\mathscr{B}^{\mathsf{opp}} \otimes \mathscr{B}) \cong \mathbf{D}(\mathscr{E}^{\mathsf{opp}} \otimes \mathscr{E})$ compatible with the tensor product and taking the diagonal bimodule $\mathscr{B}$ to the diagonal bimodule $\mathscr{E}$ and the normal bimodule $\mathscr{D}^\vee_{\mathscr{B}}$ to $\mathscr{D}^\vee_{\mathscr{E}}$. □

Our goal is to compute the normal Hochschild cohomology of $\mathscr{E}$ in $\mathscr{D}$, i.e. the derived tensor product of the diagonal bimodule $\mathscr{E}$ with $\mathscr{D}^\vee_{\mathscr{E}}$. For this we use the bar-resolution (1) of $\mathscr{E}$. Since the DG-category $\mathscr{E}$ is generated by an exceptional collection one can simplify the bar-resolution a bit.

First note that each collection $\mathbf{x}_0, \dots, \mathbf{x}_p$ of objects of $\mathscr{E}$ is just a sequence $E_{a_0}, \dots, E_{a_p}$ of exceptional objects in the collection $E_1, \dots, E_n$. Thus collections $\mathbf{x}_0, \dots, \mathbf{x}_p \in \mathscr{E}$ are in bijection with collections of integers $a_0, \dots, a_p \in \{1, \dots, n\}$. Since $\mathscr{E}(\mathbf{x}_i, \mathbf{x}_{i-1}) = \mathsf{Hom}_{\mathscr{D}}(E_{a_{i-1}}, E_{a_i})$ is acyclic when $a_i < a_{i-1}$, we can omit all collections $a_0, \dots, a_p$ which are not increasing. Moreover, since $\mathsf{Hom}_{\mathscr{D}}(E_a, E_a)$ is quasiisomorphic to the base field k and so the multiplication map

$$\mathsf{Hom}_{\mathscr{D}}(E_a, E_a) \otimes \mathsf{Hom}_{\mathscr{D}}(E_a, E_{a'}) \to \mathsf{Hom}_{\mathscr{D}}(E_a, E_{a'})$$

is a quasiisomorphism, we can omit all collections $a_0, \dots, a_p$ which are not strictly increasing. Thus we have the following

Lemma 3.6. *The diagonal bimodule $\mathscr{E}$ is quasiisomorphic to the following reduced bar-complex*

$$\bar{\mathbf{B}}(\mathscr{E}) = \bigoplus_{1\le a_0<\dots<a_p\le n} \mathsf{Hom}_{\mathscr{D}}(E_{a_p},-)\otimes \mathsf{Hom}_{\mathscr{D}}(E_{a_{p-1}},E_{a_p})\otimes\dots\otimes \mathsf{Hom}_{\mathscr{D}}(E_{a_0},E_{a_1})\otimes \mathsf{Hom}_{\mathscr{D}}(-,E_{a_0})[p],$$

Using the reduced bar-resolution we easily deduce

Proposition 3.7. *The normal Hochschild cohomology $\mathsf{NHH}^\bullet(\mathscr{E},\mathscr{D})$ of the DG-subcategory $\mathscr{E}\subset\mathscr{D}$ generated by an exceptional collection $E_1,\dots,E_n$ is isomorphic to the cohomology of the bicomplex $\mathscr{H}^{\bullet,\bullet}$ with*

$$\mathscr{H}^{-p,q} = \bigoplus_{\substack{1\le a_0<\dots<a_p\le n\\ k_0+\dots+k_p=q}} \mathsf{Hom}^{k_0}_{\mathscr{D}}(E_{a_0},E_{a_1})\otimes\dots\otimes \mathsf{Hom}^{k_{p-1}}_{\mathscr{D}}(E_{a_{p-1}},E_{a_p})\otimes \mathsf{Hom}^{k_p}_{\mathscr{D}}(E_{a_p},S^{-1}(E_{a_0})),$$

and with the differential $\mathrm{d}=\mathrm{d}'+\mathrm{d}''$, where $\mathrm{d}' : \mathscr{H}^{-p,q}\to\mathscr{H}^{-p,q+1}$ is induced by the differentials of the complexes $\mathsf{Hom}_{\mathscr{D}}$ and $\mathrm{d}'' : \mathscr{H}^{-p,q}\to\mathscr{H}^{1-p,q}$ is induced by the multiplication maps between the adjacent factors and the map

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}}(E_{a_p},S^{-1}(E_{a_0}))\otimes \mathsf{Hom}_{\mathscr{D}}(E_{a_0},E_{a_1}) &\cong\\ \cong \mathsf{Hom}_{\mathscr{D}}(E_{a_p},S^{-1}(E_{a_0}))\otimes \mathsf{Hom}_{\mathscr{D}}(S^{-1}(E_{a_0}),S^{-1}(E_{a_1})) &\to \mathsf{Hom}_{\mathscr{D}}(E_{a_p},S^{-1}(E_{a_1})).\end{aligned}$$

The bicomplex $\mathscr{H}$ will be referred to as the normal Hochschild complex of $\mathscr{E}$ in $\mathscr{D}$.

Remark 3.8. Alternatively, we could use the reduced bar-resolution of the normal bimodule

$$\bar{\mathbf{B}}(\mathscr{D}^\vee_{\mathscr{E}}) = \bigoplus_{1\le a_0<\dots<a_p\le n} \mathsf{Hom}_{\mathscr{D}}(E_{a_p},-)\otimes \mathsf{Hom}_{\mathscr{D}}(E_{a_{p-1}},E_{a_p})\otimes\dots\otimes \mathsf{Hom}_{\mathscr{D}}(E_{a_0},E_{a_1})\otimes \mathscr{D}^\vee_{\mathscr{E}}(E_{a_0},-)$$

(note that by (14) the bimodule $\mathscr{D}^\vee_{\mathscr{E}}$ is representable as a right $\mathscr{E}$-module, hence it is enough to take its bar-resolution only as of a left module). It is easy to see that tensoring it with $\mathscr{E}$ gives literally the same bicomplex $\mathscr{H}$.

Consider the spectral sequence of the bicomplex $\mathscr{H}$. Its first page is obtained by taking the cohomology with respect to d'. Thus

$$\mathbf{E}_1(\mathscr{H})^{-p,q} = \bigoplus_{\substack{1\le a_0<\dots<a_p\le n\\ k_0+\dots+k_p=q}} \mathsf{Ext}^{k_0}(E_{a_0},E_{a_1})\otimes\dots\otimes \mathsf{Ext}^{k_{p-1}}(E_{a_{p-1}},E_{a_p})\otimes \mathsf{Ext}^{k_p}(E_{a_p},S^{-1}(E_{a_0})), \tag{15}$$

The differential d_1 is induced by the multiplication maps m_2. The higher differentials d_2, d_3 and so on are induced by the higher multiplication maps m_3, m_4 and so on in the A_∞ structure on Ext's induced by the DG-structure of the complexes $\mathsf{Hom}_{\mathscr{D}}$.

The spectral sequence of the normal Hochschild complex $\mathscr{H}$ will be referred to as the normal Hochschild spectral sequence.

4. Height of an exceptional collection

4.1. **Height and pseudoheight.** The height and the pseudoheight of an exceptional collection $E_1,\dots,E_n$ are invariants controlling the coincidence of Hochschild cohomology of X and the orthogonal complement

$$\mathcal{A} = \langle E_1,\dots,E_n\rangle^\perp \tag{16}$$

of the collection.

Definition 4.1. The height of an exceptional collection $E_1,\dots,E_n$ is defined as

$$\mathrm{h}(E_1,\dots,E_n) = \min\{k\in\mathbb{Z}\mid \mathsf{NHH}^k(\mathscr{E},\mathscr{D})\neq 0\},$$

where $\mathscr{E}$ is the DG-category generated by $E_1,\dots,E_n$.

We have the following simple consequence of Theorem 3.3.

Theorem 4.2. *Let $h = \mathrm{h}(E_1, \dots, E_n)$ be the height of an exceptional collection $E_1, \dots, E_n$ and let $\mathcal{A}$ be its orthogonal complement* (16)*. The canonical restriction morphism $\mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A})$ is an isomorphism for $k \le h-2$ and a monomorphism for $k = h-1$.*

Proof. The long exact sequence of cohomology groups of the triangle (13) gives

$$\dots \to \mathsf{NHH}^k(\mathscr{E}, \mathscr{D}) \to \mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A}) \to \mathsf{NHH}^{k+1}(\mathscr{E}, \mathscr{D}) \to \dots$$

For $k \le h-2$ both extremal terms vanish, hence the middle arrow is an isomorphism. For $k = h-1$ the left term vanishes, hence the middle arrow is a monomorphism. □

Corollary 4.3. *The height of an exceptional collection is invariant under mutations.*

Proof. Mutations of the collection change neither the subcategory $\mathcal{A}$, nor its embedding into $\mathbf{D}^b(\mathsf{coh}(X))$. Therefore, the morphism $\mathsf{HH}^\bullet(X) \to \mathsf{HH}^\bullet(\mathcal{A})$ does not change. But the height is nothing but the maximal integer h for which the statement of Theorem 4.2 is true. □

The only drawback of the notion of height is that it may be difficult to compute. A priori its computation requires understanding of the higher multiplications in the Ext algebra of the exceptional collection. Below we suggest a coarser invariant, the pseudoheight, which is much easier to compute but still gives some control of the Hochschild cohomology.

For any two objects $F, F' \in \mathbf{D}(X)$ we define their relative height as

$$\mathsf{e}(F, F') = \min\{k \mid \mathsf{Ext}^k(F, F') \neq 0\}.$$

Definition 4.4. The pseudoheight of the exceptional collection $E_1, \dots, E_n$ is defined as

$$\mathrm{ph}(E_1, \dots, E_n) = \min_{1 \le a_0 < a_1 < \dots < a_p \le n} \Big(\mathsf{e}(E_{a_0}, E_{a_1}) + \dots + \mathsf{e}(E_{a_{p-1}}, E_{a_p}) + \mathsf{e}(E_{a_p}, S^{-1}(E_{a_0})) - p \Big).$$

The minimum is taken over the set of all chains of indices $1 \le a_0 < a_1 < \dots < a_p \le n$. Note that the length p of a chain enters nontrivially into the expression under the minimum.

The pseudoheight gives a lower bound for the height.

Lemma 4.5. *We have $\mathrm{h}(E_1, \dots, E_n) \ge \mathrm{ph}(E_1, \dots, E_n)$.*

Proof. The pseudoheight is the minimal total degree of nontrivial terms of the first page of the normal Hochschild spectral sequence (15). Since the spectral sequence converges to normal Hochschild cohomology $\mathsf{NHH}^\bullet(\mathscr{E}, \mathscr{D})$, we conclude that the latter is zero in degrees $k < \mathrm{ph}(E_1, \dots, E_n)$. Therefore $\mathrm{h}(E_1, \dots, E_n) \ge \mathrm{ph}(E_1, \dots, E_n)$. □

Since the pseudoheight is not greater than the height, it gives the same restriction on the morphism of Hochschild cohomology.

Corollary 4.6. *Let $h = \mathrm{ph}(E_1, \dots, E_n)$ be the pseudoheight of an exceptional collection $E_1, \dots, E_n$ and let $\mathcal{A}$ be its orthogonal complement* (16)*. The canonical restriction morphism $\mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A})$ is an isomorphism for $k \le h-2$ and a monomorphism for $k = h-1$.*

Somtimes one can easily show that the pseudoheight equals the height.

Proposition 4.7. *Assume that the pseudoheight $\mathrm{ph}(E_1, \dots, E_n)$ is achieved on a chain of length* 0*, i.e. $\mathrm{ph}(E_1, \dots, E_n) = \mathsf{e}(E_i, S^{-1}(E_i))$ for some i. Then $\mathrm{h}(E_1, \dots, E_n) = \mathrm{ph}(E_1, \dots, E_n)$.*

Proof. Let $k = \mathrm{ph}(E_1,\dots,E_n)$ and assume that $\mathsf{e}(E_i, S^{-1}(E_i)) = k$. Note that we have an inclusion $\mathsf{Ext}^k(E_i, S^{-1}(E_i)) \subset \mathbf{E}_1(\mathscr{H})^{0,k}$, hence all the higher differentials of the normal Hochschild spectral sequence vanish on this space (just because the higher differentials increase p and $\mathscr{H}$ is concentrated in nonpositive degrees with respect to p). On the other hand, all differentials increase the total degree $p+q$ and by assumption k is the minimal total degree of nonzero elements of the first page of the spectral sequence. Hence no nontrivial differentials of the spectral sequence have target at $\mathsf{Ext}^k(E_i, S^{-1}(E_i))$, hence it embedds into $\mathsf{NHH}^k(\mathscr{E},\mathscr{D})$. Thus $\mathrm{h}(E_1,\dots,E_n) \le k = \mathrm{ph}(E_1,\dots,E_n)$. On the other hand we know that $\mathrm{h}(E_1,\dots,E_n) \ge \mathrm{ph}(E_1,\dots,E_n)$ by Lemma 4.5. Thus $\mathrm{h}(E_1,\dots,E_n) = \mathrm{ph}(E_1,\dots,E_n)$. ☐

4.2. **Formal deformation spaces.** We refer to [KS] for generalities about deformation theory of A_∞-algebras and categories. Recall that the second Hochschild cohomology is the tangent space to the deformation space of a category and the third Hochschild cohomology is the space of obstructions. By Theorem 4.2 if $\mathrm{h}(E_1,\dots,E_n) \ge 4$ then the map $\mathsf{HH}^2(X) \to \mathsf{HH}^2(\mathcal{A})$ is an isomorphism and the map $\mathsf{HH}^3(X) \to \mathsf{HH}^3(\mathcal{A})$ is a monomorphism. As a consequence we have

Proposition 4.8. *If $\mathrm{h}(E_1,\dots,E_n) \ge 4$ then the formal deformation spaces of categories $\mathbf{D}(X)$ and $\mathcal{A}$ are isomorphic.*

Proof. The formal deformation space of a DG-category $\mathscr{D}$ is described in terms of the Gerstenhaber algebra structure on the Hochschild cohomology of $\mathscr{D}$. To be more precise only the cohomology in degrees up to 3 play role. Now let $\mathscr{D}$ be the Čech enhancement of $\mathbf{D}(X)$ and $\mathscr{A}$ the induced enhancement of $\mathcal{A}$. The restriction morphism $\mathsf{HH}^\bullet(\mathscr{D}) \to \mathsf{HH}^\bullet(\mathscr{A})$ is a morphism of Gerstenhaber algebras (this is clear from the explicit formulas for the multiplication and the bracket) which is an isomorphism in degrees up to 2 and a monomorphism in degree 3. Therefore the formal deformation spaces are isomorphic. ☐

4.3. **Anticanonical pseudoheight.** Sometimes it is more convenient to replace the inverse Serre functor in the definition of the height and pseudoheight by the anticanonical twist. Of course, this just shifts the result by $\dim X$.

Definition 4.9. The anticanonical pseudoheight of an exceptional collection $E_1,\dots,E_n$ is defined as

$$\mathrm{ph}_{\mathrm{ac}}(E_1,\dots,E_n) = \min_{1\le a_0<a_1<\dots<a_p\le n} \Big(\mathsf{e}(E_{a_0},E_{a_1}) + \dots + \mathsf{e}(E_{a_{p-1}},E_{a_p}) + \mathsf{e}(E_{a_p}, E_{a_0}\otimes\omega_X^{-1}) - p\Big).$$

Thus $\mathrm{ph}_{\mathrm{ac}} = \mathrm{ph} - \dim X$. The anticanonical height is defined as $\mathrm{h}_{\mathrm{ac}}(E_1,\dots,E_n) = \mathrm{h}(E_1,\dots,E_n) - \dim X$.

Let $E_1,\dots,E_n$ be an exceptional collection. The collection

$$E_1,\dots,E_n, E_1\otimes\omega_X^{-1},\dots,E_n\otimes\omega_X^{-1}$$

will be called the (anticanonically) extended collection. We will say that the extended collection is Hom-free if $\mathsf{Ext}^p(E_i,E_j) = 0$ for $p \le 0$ and all $1 \le i < j \le i+n$. A Hom-free collection is called cyclically Ext^1-connected if there is a chain $1 \le a_0 < a_1 < \dots < a_{p-1} < a_p \le n$ such that $\mathsf{Ext}^1(E_{a_s}, E_{a_{s+1}}) \ne 0$ for all $s = 0,1,\dots,p-1$ and $\mathsf{Ext}^1(E_{a_p}, E_{a_0}\otimes\omega_X^{-1}) \ne 0$.

Lemma 4.10. *If $E_1,\dots,E_n$ is an exceptional collection such that the anticanonically extended collection is* Hom*-free then $\mathrm{ph}_{\mathrm{ac}}(E_1,\dots,E_n) \ge 1$ and $\mathrm{ph}(E_1,\dots,E_n) \ge 1+\dim X$. If, in addition, the extended collection is not cyclically* Ext^1*-connected then $\mathrm{ph}_{\mathrm{ac}}(E_1,\dots,E_n) \ge 2$ and $\mathrm{ph}(E_1,\dots,E_n) \ge 2+\dim X$.*

Proof. If the extended collection is Hom-free then $\mathsf{e}(E_{a_i},E_{a_{i+1}}) \ge 1$ and $\mathsf{e}(E_{a_p}, E_{a_0}\otimes\omega_X^{-1}) \ge 1$, hence

$$\mathsf{e}(E_{a_0},E_{a_1}) + \dots + \mathsf{e}(E_{a_{p-1}},E_{a_p}) + \mathsf{e}(E_{a_p}, E_{a_0}\otimes\omega_X^{-1}) - p \ge (p+1) - p = 1,$$

hence the anticanonical pseudoheight is not less than 1. If, moreover, the extended collection is not cyclically Ext^1-connected then in the sum $\mathsf{e}(E_{a_0},E_{a_1}) + \dots + \mathsf{e}(E_{a_{p-1}},E_{a_p}) + \mathsf{e}(E_{a_p}, E_{a_0}\otimes\omega_X^{-1})$ at least

one summand is at least 2, hence the sum is at least $p+2$, hence the LHS above is at least 2, hence the anticanonical pseudoheight is not less than 2. $\square$

5. Examples

In this section we provide several examples which show that the anticanonical height is easily computable. All these examples deal with quasiphantom categories constructed recently in the derived categories of some surfaces of general type. In all examples below we will use the following simple observation.

Lemma 5.1. *Let X be a surface with ample canonical class K_X and (L_1, L_2) a pair of line bundles. If*

$$c_1(L_1) \cdot K_X \geq c_1(L_2) \cdot K_X,$$

then $\mathsf{Hom}(L_1, L_2) = 0$ unless $L_1 \cong L_2$.

Proof. A nonzero morphism $L_1 \to L_2$ gives a section of the line bundle $L_1^{-1} \otimes L_2$, hence $L_1^{-1} \otimes L_2 \cong \mathcal{O}_X(D)$ for an effective divisor D. Since K_X is ample we have $D \cdot K_X > 0$ unless $D = 0$. This proves the Lemma. $\square$

Another useful observation is the following

Lemma 5.2. *Let X be a smooth projective surface with $H^2(X, \omega_X^{-1}) \neq 0$. If $E_1, \dots, E_n$ is an exceptional collection consisting of line bundles then $\mathrm{ph_{ac}}(E_1, \dots, E_n) \leq 2$. Moreover, if $\mathrm{ph_{ac}}(E_1, \dots, E_n) = 2$ then $\mathrm{h_{ac}}(E_1, \dots, E_n) = 2$ as well.*

Proof. Since E_i is a line bundle we have $\mathsf{Ext}^k(E_i, E_i \otimes \omega_X^{-1}) = H^k(X, E_i^\vee \otimes E_i \otimes \omega_X^{-1}) = H^k(X, \omega_X^{-1})$ hence $\mathrm{ph_{ac}}(E_1, \dots, E_n) \leq \mathsf{e}(E_i, E_i \otimes \omega_X^{-1}) \leq 2$. If the anticanonical pseudoheight is 2 then by Proposition 4.7 the anticanonical height is also 2. $\square$

5.1. **Burniat surfaces.** Alexeev and Orlov in [AO] have constructed an exceptional collection of length 6 in the derived category of a Burniat surface (Burniat surfaces is a family of surfaces of general type with $p_g = q = 0$ and $K^2 = 6$). The orthogonal subcategory $\mathcal{A} \subset \mathbf{D}(X)$ has trivial Hochschild homology and $K_0(\mathcal{A}) = (\mathbb{Z}/2\mathbb{Z})^6$.

Proposition 5.3. *The height of the Alexeev–Orlov exceptional collection is* 4.

Proof. The collection consists of line bundles, the canonical degrees of the anticanonically extended collection are given by the following sequence

$$3, 3, 2, 2, 2, 0 \ ; \ -3, -3, -4, -4, -4, -6,$$

the semicolon separates the extended part. In particular, the degrees do not increase, so by Lemma 5.1 the collection is Hom-free. Therefore $\mathrm{ph_{ac}}(E_1, \dots, E_6) \geq 1$ by Lemma 4.10. Let us also check that the collection is not cyclically Ext^1-connected. In other words for any chain $1 \leq a_0 < \dots < a_p \leq 6$ we have to check that either at least for one i we have $\mathsf{Ext}^1(E_{a_i}, E_{a_{i+1}}) = 0$ or $\mathsf{Ext}^1(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) = 0$. If $p > 0$ then $\mathsf{Ext}^1(E_{a_0}, E_{a_1}) = 0$ by Lemma 4.8 of [AO]. If $p = 0$ then since E_{a_0} is a line bundle

$$\mathsf{Ext}^1(E_{a_0}, E_{a_0} \otimes \omega_X^{-1}) = H^1(X, \omega_X^{-1})$$

and it is easy to see that it is also 0. Thus $\mathrm{ph_{ac}}(E_1, \dots, E_6) \geq 2$ by Lemma 4.10. By Lemma 5.2 we conclude that $\mathrm{ph_{ac}}(E_1, \dots, E_6) = \mathrm{h_{ac}}(E_1, \dots, E_6) = 2$. $\square$

Corollary 5.4. *If X is a Burniat surface then the natural restriction morphism $\mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A})$ is an isomorphism for $k \leq 2$ and a monomorphism for $k = 3$. In particular, the formal deformation space of $\mathcal{A}$ is isomorphic to that of $\mathbf{D}(X)$.*

5.2. **The Beauville surface.** Galkin and Shinder in [GS] have constructed six different exceptional collections of length 4 in the derived category of the Beauville surface (Beauville surface is an example of a surface of general type with $p_g = q = 0$ and $K^2 = 8$). The orthogonal subcategory $\mathcal{A} \subset \mathbf{D}(X)$ to any of those has trivial Hochschild homology and $K_0(\mathcal{A}) = (\mathbb{Z}/5\mathbb{Z})^2$. We will consider the first two exceptional collections (called I_1 and I_0 in Theorem 3.5 of loc. cit.), since the other four can be obtained from them by taking the anticanonically extended collection and then restricting to its length 4 subcollections (which are automatically exceptional due to Serre duality). This operation clearly does not change the pseudoheight.

Proposition 5.5. *The pseudoheight of the collection I_1 is 4 and of the collection I_0 is 3. The height of both collections is 4.*

Proof. The collections consist of line bundles, the canonical degrees of the anticanonically extended collections are given by the following sequences

$$\begin{array}{ll} I_1 : & 0, -2, -2, -4 \ ; \ -8, -10, -10, -12 \\ I_0 : & 0, -2, -4, -6 \ ; \ -8, -10, -12, -14 \end{array}$$

The degrees do not increase, so by Lemma 5.1 the collections are Hom-free. Thus $\mathrm{ph}_{\mathrm{ac}}(E_1, \dots, E_4) \geq 1$ by Lemma 4.10. Let us check whether the collections are Ext^1-connected. It turns out that the collection I_0 is Ext^1-connected by the chain $a = (1, 2, 3, 4)$, hence $\mathrm{ph}_{\mathrm{ac}}(I_0) = 1$, while the collection I_1 is not Ext^1-connected, since there is no Ext^1 from $E_1, \dots, E_4$ to $E_1 \otimes \omega_X^{-1}, \dots, E_4 \otimes \omega_X^{-1}$ — this can be easily seen from the character matrices in Proposition 3.7 of loc. cit., hence $\mathrm{ph}_{\mathrm{ac}}(I_1) = 2$. Again using Lemma 5.2 we conclude that $\mathrm{h}_{\mathrm{ac}}(I_1) = 2$.

To show that the anticanonical height of the collection I_0 is also 2, or equivalently that its height is 4, we need to look at the normal Hochschild spectral sequence. It is clear that the only component of total degree 3 in $\mathbf{E}_1(\mathscr{H})$ is

$$\mathsf{Ext}^1(E_1, E_2) \otimes \mathsf{Ext}^1(E_2, E_3) \otimes \mathsf{Ext}^1(E_3, E_4) \otimes \mathsf{Ext}^3(E_4, S^{-1}(E_1)) \subset \mathbf{E}_1(\mathscr{H})^{-3,6}.$$

The differential d_1 from this term lands into the direct sum of the following four terms

$$\begin{array}{l} \mathsf{Ext}^2(E_1, E_3) \otimes \mathsf{Ext}^1(E_3, E_4) \otimes \mathsf{Ext}^3(E_4, S^{-1}(E_1)), \\ \mathsf{Ext}^1(E_1, E_2) \otimes \mathsf{Ext}^2(E_2, E_4) \otimes \mathsf{Ext}^3(E_4, S^{-1}(E_1)), \\ \mathsf{Ext}^1(E_1, E_2) \otimes \mathsf{Ext}^1(E_2, E_3) \otimes \mathsf{Ext}^4(E_3, S^{-1}(E_1)), \\ \mathsf{Ext}^1(E_2, E_3) \otimes \mathsf{Ext}^1(E_3, E_4) \otimes \mathsf{Ext}^4(E_4, S^{-1}(E_2)), \end{array}$$

which is a subspace of $\mathbf{E}_1(\mathscr{H})^{-2,6}$. The map into each term is just the multiplication. It is easy to check that even the first map $\mathsf{Ext}^1(E_1, E_2) \otimes \mathsf{Ext}^1(E_2, E_3) \to \mathsf{Ext}^2(E_1, E_3)$ is nonzero, hence $\mathbf{E}_2(\mathscr{H})^{-3,6} = 0$ and it follows that $\mathrm{h}(E_1, \dots, E_4) \geq 4$. Finally, using the argument of Proposition 4.7 it is easy to see that $\mathbf{E}_\infty(\mathscr{H})^{0,4} \neq 0$, so $\mathrm{h}(E_1, \dots, E_4) = 4$. □

Corollary 5.6. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the orthogonal subcategories of the exceptional collections I_0 and I_1 on the Beauville surface. Then the natural restriction morphisms $\mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A}_t)$ are isomorphisms for $k \leq 2$ and monomorphisms for $k = 3$.*

In fact the Hochschild cohomology of the Beauville surface is very simple (see Lemma 2.6 of [GS])

$$\mathsf{HH}^0(X) = \mathsf{k}, \quad \mathsf{HH}^1(X) = \mathsf{HH}^2(X) = 0, \quad \mathsf{HH}^3(X) = \mathsf{k}^6, \quad \mathsf{HH}^4(X) = \mathsf{k}^9.$$

It follows that $\mathsf{HH}^0(\mathcal{A}_{0,1}) = \mathsf{k}$ and $\mathsf{HH}^1(\mathcal{A}_{0,1}) = 0$, $\mathsf{HH}^2(\mathcal{A}_{0,1}) = 0$. It is an interesting question whether the quasiphantom categoies $\mathcal{A}_0$ and $\mathcal{A}_1$ are equivalent.

5.3. **The classical Godeaux surface.** Böhning–Graf von Bothmer–Sosna in [BBS] have constructed an exceptional collection of line bundles of length $n = 11$ in the derived category of the classical Godeaux surface. The orthogonal subcategory $\mathcal{A} \subset \mathbf{D}(X)$ has trivial Hochschild homology and $K_0(\mathcal{A}) = \mathbb{Z}/5\mathbb{Z}$. Historically, this is the first example of a quasiphantom category. However, it is more complicated than the other two examples, which is the reason why we considered those first. To find the height of the Böhning–Graf von Bothmer–Sosna collection we will need the following

Lemma 5.7. *If $E_1, \dots, E_{11}$ is the Böhning–Graf von Bothmer–Sosna collection then*

$$\mathsf{Hom}(E_3, E_2(-K)) = \mathsf{Ext}^1(E_3, E_2(-K)) = 0 \quad \text{and} \quad \mathsf{Ext}^2(E_3, E_2(-K)) \neq 0.$$

Proof. By Serre duality we should compute $\mathsf{Ext}^p(E_2, E_3(2K)) = H^p(X, E_2^{-1} \otimes E_3(2K))$. Recall that $\mathsf{Hom}(E_2, E_3) \neq 0$, hence $E_2^{-1} \otimes E_3 \cong \mathcal{O}_X(D)$ for some effective divisor D. Moreover, by Lemma 8.2 and 10.2 of [BBS] we have $D \cdot K_X = 1$ and $\chi(\mathcal{O}_X(D)) = -1$, hence from the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$$

we conclude that $\chi(\mathcal{O}_D(D)) = -2$, so it follows from the results of section 4 of [BBS] that D is one of 15 lines on X. In paticular, $D \cong \mathbb{P}^1$ and $\mathcal{O}_D(D) = \mathcal{O}_D(-3)$. Twisting the above sequence by $2K$ we obtain

$$0 \to \mathcal{O}_X(2K) \to \mathcal{O}_X(D + 2K) \to \mathcal{O}_D(-1) \to 0.$$

Since $\mathcal{O}_X(2K)$ has only H^0 which is of dimension 2 and $\mathcal{O}_D(-1)$ has no cohomology at all, we conclude that $\mathcal{O}_X(D + 2K)$ has only H^0 which is also of dimension 2. $\square$

Proposition 5.8. *The pseudoheight of the Böhning–Graf von Bothmer–Sosna exceptional collection is* 3 *and its height is* 4.

Proof. By Lemma 10.2 of [BBS] and Lemma 5.7 above we have

$$\mathsf{e}(E_2, E_3) + \mathsf{e}(E_3, E_2(-K)) - 1 = 0 + 2 - 1 = 1,$$

hence $\mathrm{ph}_{\mathrm{ac}}(E_1, \dots, E_{11}) \leq 1$. So, we only have to check that $\mathrm{ph}_{\mathrm{ac}}(E_1, \dots, E_{11}) > 0$.

Note that the extended collection is almost Hom-free. Indeed, all objects of the anticanonically extended collection are line bundles of the following canonical degrees

$$0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0 \ ; \ -1, -1, 0, -1, -1, -1, 0, -1, -1, -1, -1.$$

By Lemma 5.1 we could only have morphisms between the objects of a nonextended collection (or their twists), which by [BBS, Lemma 10.2] are nonzero only from E_2 to E_3.

Therefore, to show that the anticanonical pseudoheight is positive we only have to check that there are no chains $1 \leq a_0 < \dots < a_p \leq 11$ such that in the chain of line bundles $E_{a_0}, \dots, E_{a_p}, E_{a_0}(-K)$ there is an adjacent pair which has Hom, and all other adjacent pairs have Ext^1 between them.

Assume such a chain exists. The arguments above show that the pair (E_2, E_3) should be in the chain. In particular we should have $a_0 \leq 2$. Assume first that $a_0 = 1$. Then we must have $a_1 = 2$, $a_2 = 3$. But $\mathsf{Ext}^1(E_1, E_2) = 0$, so this a contradiction. Assume that $a_0 = 2$. Then $a_1 = 3$. Note that from E_3 to $E_4, \dots, E_{11}$ there are no Ext^1, hence p should be 1. But $\mathsf{Ext}^1(E_3, E_2(-K)) = 0$ by Lemma 5.7, so this is again a contradiction. Thus the anticanonical pseudoheight is positive and we are done.

To compute the height we have to look at the spectral sequence. By the arguments above we already know that $\mathrm{h}(E_1, \dots, E_{11}) \geq \mathrm{ph}(E_1, \dots, E_{11}) = \mathrm{ph}_{\mathrm{ac}}(E_1, \dots, E_{11}) + 2 = 3$. On the other hand, by the arguments of Proposition 4.7 it is easy to show that $\mathrm{h}(E_1, \dots, E_{11}) \leq 4$. So, the only thing to check is whether the Hochschild homology of the normal bimodule is nontrivial in degree 3. For this one has to analyze all chains $a_0 < \dots < a_p$ on which the pseudoheight is achieved and to analyze the products (and

maybe the higher products) of their terms. We already know that the pseudoheight is achieved on the chain $(2,3)$. But the argument of Lemma 5.1 shows that the composition

$$\mathsf{Hom}(E_2, E_3) \otimes \mathsf{Ext}^2(E_3, E_2(-K)) \to \mathsf{Ext}^2(E_2, E_2(-K))$$

is a monomorphism, hence this term is killed in the second page of the spectral sequence. On the other hand, the only other chains on which the pseudoheight might be achieved are

$$(1,3), (1,7), (2,7), (4,7), (5,7), (6,7).$$

An explicit computation [BB] shows that this is not the case for all of them, hence the height is 4. □

Corollary 5.9. *The natural restriction morphism* $\mathsf{HH}^k(X) \to \mathsf{HH}^k(\mathcal{A})$ *is an isomorphism for* $k \le 2$ *and a monomorphism for* $k = 3$ *for the classical Godeaux surface. In particular, the formal deformation space of* $\mathcal{A}$ *is isomorphic to that of* $\mathbf{D}(X)$*.*

6. The necessary and sufficient conditions of fullness

Quite unexpectedly, the height gives a necessary condition of fullness of an exceptional collection.

Proposition 6.1. *Let* X *be a smooth projective variety and let* $E_1, \dots, E_n$ *be an exceptional collection in* $\mathbf{D}^b(\mathsf{coh}(X))$*. If* $\mathrm{h}(E_1, \dots, E_n) > 0$ *then the collection is not full.*

Proof. Let $\mathcal{A}$ be the orthogonal subcategory. If $\mathrm{h}(E_1, \dots, E_n) > 0$ then by Theorem 4.2 the morphism $\mathsf{HH}^0(X) \to \mathsf{HH}^0(\mathcal{A})$ is a monomorphism. Since $\mathsf{HH}^0(X) = H^0(X, \mathcal{O}_X) \neq 0$, we conclude that $\mathsf{HH}^0(\mathcal{A}) \neq 0$. Therefore $\mathcal{A} \neq 0$ so the collection is not full. □

Since the height is not smaller than the pseudoheight we obtain also an easily verifiable criterion of nonfullness.

Corollary 6.2. *If* $\mathrm{ph}(E_1, \dots, E_n) > 0$ *then the collection is not full.*

In particular, it follows that in all examples of section 5 the collections are not full. Note that this argument does not use the computation of the Grothendieck group of these categories and can be applied also for real phantom categories, when the Grothendieck group does not help.

Even more unexpectedly is that one can also use the normal Hochschild cohomology to deduce the fullness of the collection. Let

$$\rho : \mathsf{NHH}^\bullet(\mathscr{E}, \mathscr{D}) \to \mathsf{HH}^\bullet(X)$$

be the morphism of (13).

Theorem 6.3. *Let* X *be a smooth and connected projective variety and let* $E_1, \dots, E_n$ *be an exceptional collection in* $\mathbf{D}^b(\mathsf{coh}(X))$*. Assume that there is an element* $\xi \in \mathsf{NHH}^0(\mathscr{E}, \mathscr{D})$ *such that* $\rho(\xi) \neq 0$*. Then the collection* $E_1, \dots, E_n$ *is full.*

Proof. Again, let $\mathcal{A}$ be the orthogonal subcategory. Note that $\mathsf{HH}^0(X) = H^0(X, \mathcal{O}_X) = \mathsf{k}$ since X is connected. Therefore, after rescaling we can assume that $\rho(\xi) = 1_X \in \mathsf{HH}^0(X)$. Since (13) is exact, the image of 1_X under the restriction morphism $\mathsf{HH}^0(X) \to \mathsf{HH}^0(\mathcal{A})$ is zero. On the other hand, it is clear that the image equals $1_\mathcal{A} \in \mathsf{HH}^0(\mathcal{A})$. We conclude that $1_\mathcal{A} = 0$ in $\mathsf{HH}^\bullet(\mathcal{A})$. But this means that $\mathcal{A} = 0$, hence the collection is full. □

Of course, to use this criterion one needs a method to check that $\rho(\xi) \neq 0$. Note that for each object $E \in \mathbf{D}^b(\mathsf{coh}(X))$ there is a canonical evaluation morphism

$$\mathsf{ev}_E : \mathsf{HH}^0(X) \to \mathsf{Hom}(E, E).$$

For connected X we know that $\mathsf{HH}^0(X) = \mathsf{k}$ and $\mathsf{ev}_E(\lambda) = \lambda\cdot\mathsf{id}_E \in \mathsf{Hom}(E,E)$. So to check that $\rho(\xi) \ne 0$ it is enough to find an object E such that $\mathsf{ev}_E(\rho(\xi)) \ne 0$. It is quite natural to take for E one of the objects E_i.

Let $\eta_i \in \mathsf{Hom}(S^{-1}(E_i), E_i)$ be the generator of

$$\mathsf{Hom}(S^{-1}(E_i), E_i) \cong \mathsf{Hom}(E_i, E_i)^\vee = \mathsf{k}^\vee = \mathsf{k}.$$

Recall that the normal Hochschild cohomology $\mathsf{NHH}^\bullet(\mathscr{E},\mathscr{D})$ is computed by the spectral sequence $\mathbf{E}_r(\mathscr{H})^{p,q}$. In particular, it follows that $\mathsf{NHH}^0(\mathscr{E},\mathscr{D})$ has a filtration with factors $\mathbf{E}_\infty(\mathscr{H})^{-p,p}$. Thus $\mathbf{E}_\infty(\mathscr{H})^{0,0}$ is a subspace in $\mathsf{NHH}^0(\mathscr{E},\mathscr{D})$ and $\mathbf{E}_\infty(\mathscr{H})^{1-n,n-1}$ is the quotient of $\mathsf{NHH}^0(\mathscr{E},\mathscr{D})$.

Proposition 6.4. *Let p be the maximal integer such that $\mathbf{E}_\infty(\mathscr{H})^{-p,p} \ne 0$. Let*

$$\xi_p = \sum \xi_p^{a,k} \in \bigoplus_{a,k} \mathsf{Ext}^{k_0}(E_{a_0}, E_{a_1}) \otimes \cdots \otimes \mathsf{Ext}^{k_{p-1}}(E_{a_{p-1}}, E_{a_p}) \otimes \mathsf{Ext}^{k_p}(E_{a_p}, S^{-1}(E_{a_0})) \subset \mathbf{E}_1(\mathscr{H})^{-p,p}$$

(the sum is over the set of all chains $a = (1 \le a_0 < \cdots < a_p \le n)$ of length $p+1$ and over all collections of integers $k = (k_0,\dots,k_p)$ such that $k_0 + \cdots + k_p = p$) be a lift of a nontrivial element $\xi \in \mathbf{E}_\infty(\mathscr{H})^{-p,p}$. Let ξ_p^i be the sum of those $\xi_p^{a,k}$ for which $a_0 = i$. Then up to a nonzero constant

$$\mathsf{ev}_{E_i}(\rho(\xi)) = m_{p+2}(\xi_p^i \otimes \eta_i).$$

In particular, if $m_{p+2}(\xi_p^i \otimes \eta_{a_0}) \ne 0$ for some i, then the collection $E_1,\dots,E_n$ is full.

Proof. Evaluation morphism is the composition

$$\mathsf{HH}^\bullet(X) \cong \mathsf{Ext}^\bullet(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \to \mathsf{Ext}^\bullet(E \boxtimes E^\vee, \Delta_*\mathcal{O}_X) \cong$$
$$\cong \mathsf{Ext}^\bullet(\mathbb{L}\Delta^*(E \boxtimes E^\vee), \mathcal{O}_X) \cong \mathsf{Ext}^\bullet(E \overset{\mathbb{L}}{\otimes} E^\vee, \mathcal{O}_X) \cong \mathsf{Ext}^\bullet(E,E).$$

where the first morphism is induced by the natural map $\mathsf{tr}_E : E \boxtimes E^\vee \to \Delta_*\mathcal{O}_X$ (which corresponds under the above chain of morphisms to $\mathsf{id}_E \in \mathsf{Hom}(E,E)$). On the other hand, since $\Delta_*\mathcal{O}_X$ is a perfect complex on $X \times X$ we have an isomorphism

$$\mathsf{Ext}^\bullet(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \cong \mathbf{H}^\bullet(X \times X, \Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} (\Delta_*\mathcal{O}_X)^\vee),$$

and by Grothendieck duality we have

$$(\Delta_*\mathcal{O}_X)^\vee \cong \Delta_*\Delta^!\mathcal{O}_{X\times X} \cong \Delta_*\omega_X^{-1}[-\dim X].$$

Dualizing the morphism tr_E we obtain a morphism

$$\eta_E : \Delta_*\omega_X^{-1}[-\dim X] \to E^\vee \boxtimes E$$

which combined with a sequence of isomorphisms

$$\mathbf{H}^\bullet(X \times X, \Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} (E^\vee \boxtimes E)) \cong \mathbf{H}^\bullet(X, \mathbb{L}\Delta^*(E^\vee \boxtimes E)) \cong \mathbf{H}^\bullet(X, E^\vee \overset{\mathbb{L}}{\otimes} E) \cong \mathsf{Ext}^\bullet(E,E)$$

gives a commutative diagram

$$\begin{tikzcd}
\mathsf{Ext}^\bullet(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \arrow[r, "\cong"] \arrow[d, "\mathsf{ev}_E"'] & \mathbf{H}^\bullet(X \times X, \Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} (\Delta_*\mathcal{O}_X)^\vee) \arrow[d, "\eta_E"] \\
\mathsf{Ext}^\bullet(E,E) \arrow[r, equal] & \mathsf{Ext}^\bullet(E,E)
\end{tikzcd}$$

Translating this into the derived category $\mathbf{D}(\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D})$ of $\mathscr{D}$-$\mathscr{D}$-bimodules via the equivalence μ we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{HH}^\bullet(\mathscr{D}) \arrow[r, "\cong"] \arrow[d, "\mathsf{ev}_{\mathbf{x}}"'] & \mathscr{D} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}} \mathscr{D}^\vee \arrow[d, "1\otimes\eta_{\mathbf{x}}"] \\ \mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{x}) \arrow[r, equal] & \mathsf{Hom}_{\mathscr{D}}(\mathbf{x},\mathbf{x}) \end{tikzcd}$$

where $\eta_{\mathbf{x}} : \mathscr{D}^\vee \to h_{\mathbf{x}} \otimes h^{\mathbf{x}}$ is the canonical morphism.

Thus we have to identify the morphism $\eta_{\mathbf{x}}$ as a morphism from the (reduced) bar-resolution of $\mathscr{D}^\vee$, restrict it to the bar-resolution of $\mathscr{D}^\vee_{\mathscr{E}}$ and then tensor it with $\mathscr{E}$. This turns out to be a difficult question, and we are not able to write down a general answer. However, we will write down a simpler morphism which is enough for our purposes.

Assume that $\epsilon(\mathbf{x}) = E$ is an exceptional object. Then in $\mathbf{D}(\mathscr{D}^{\mathsf{opp}}\otimes\mathscr{D}) \cong \mathbf{D}(X\times X)$ there is a unique morphism from $\mathscr{D}^\vee$ to $h_{\mathbf{x}}\otimes h^{\mathbf{x}}$. This morphism restricts to a nontrivial morphism

$$\mathscr{D}^\vee(\mathbf{x},-) \xrightarrow{\eta_{\mathbf{x}}} (h_{\mathbf{x}}\otimes h^{\mathbf{x}})(\mathbf{x},-) = \mathsf{Hom}_{\mathscr{D}}(E,E)\otimes h^{\mathbf{x}} \cong h^{\mathbf{x}}.$$

By definition of the bimodule $\mathscr{D}^\vee$ the image under ϵ of the above composition is a morphism $S^{-1}(E)\to E$ which is unique as E is exceptional. On the other hand, let $\bar{\eta}_E$ be a closed element of degree 0 in $\mathsf{Hom}_{\mathscr{D}}(S^{-1}(E),E)$. By definition of $\mathscr{D}^\vee$ the multiplication by $\bar{\eta}_E$ defines a nontrivial map $\mathscr{D}^\vee(\mathbf{x},-)\to h^{\mathbf{x}}$ which thus has to be equal to the map η_E. Thus we have showed that the composition $\mathsf{ev}_E\circ\rho$ coincides up to a nonzero constant with the map induced by the $\bar{\eta}_E$ multiplication. It follows that in the spectral sequence the induced map is the map $m_{p+2}(-\otimes\eta_E)$. □

As an example consider $X=\mathbb{P}^{n-1}$, the projective space, and $(E_1,\dots,E_n) = (\mathcal{O},\dots,\mathcal{O}(n-1))$. Then $\mathsf{Ext}^k(E_i,E_j)\neq 0$ only for $k=0$ and $\mathsf{Ext}^k(E_j,S^{-1}(E_i)) = \mathsf{Ext}^{k+1-n}(E_j,E_i(n)) \neq 0$ only for $k=n-1$. Thus $\mathbf{E}_1(\mathscr{H})^{-p,q}\neq 0$ only for $q=n-1$. Therefore the only nontrivial differential in the spectral sequence d_1 is given by the multiplication m_2, the spectral sequence degenerates in the second term, and so $\mathsf{NHH}^\bullet(\mathscr{E},\mathscr{D})$ is the cohomology of the complex

$$V^{\otimes n} \to \bigoplus_{i=1}^{n} V^{\otimes i-1}\otimes S^2V\otimes V^{\otimes n-i-1} \to \dots$$

(the last summand in the second term corresponds to symmetrization of the last and the first factors of the first term). Therefore, $\mathsf{NHH}^0(\mathscr{E},\mathscr{D})$ is the set of all $\xi\in V^{\otimes n}$ such that the symmetrization of ξ with respect to any pair of cyclically adjacent indices is zero. Thus ξ should be completely antisymmetric, that is $\xi\in\Lambda^nV\subset V^{\otimes n}$. One can check that $m_{n+1}(\xi\otimes\eta_1)\neq 0$ which gives yet another proof of the fact that the Beilinson collection is full.

The element ξ looks to be closely related to the *quantum determinant* considered in [BP]. It would be very interesting to understand the relationship.

An advantage of this criterion of fullness is the fact that to check fullness of a collection one just has to guess appropriate ξ. After that only two things should be checked. First, that ξ is a cocylcle, which means that multiplications $m_2,\dots,m_p$ vanish on ξ, and second, that $m_{p+1}(\xi\otimes\eta_1)$ does not vanish.

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